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Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function and the AC-Laplace Transform

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Authors' contributions

After authors TM and KS published a paper on the solution of an inhomogeneous differential equation in terms of the Green's function and distribution theory, author TM began writing a manuscript on the same theme in terms of the AC-Laplace transform, that is an extension of the Laplace transform, instead of distribution theory. Both authors collaborated to complete the manuscript. Both authors read and approved the final manuscript.

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Abstract

The particular solutions of inhomogeneous differential equations with polynomial coefficients in terms of the Green's function are obtained in the framework of the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation. In particular, discussions are given on Kummer's and the hypergeometric differential equation, and also on a fractional differential equation with coefficients of polynomial of at most first degree.

Keywords: Green's function; AC-Laplace transform; particular solution; Kummer's differential equation; hypergeometric differential equation; fractional differential equation.

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1 Introduction

In [1, 2], for Kummer's and hypergeometric differential equations, complementary solutions expressed by the confluent hypergeometric series and the hypergeometric series, respectively, are obtained, by using the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation, distribution theory and the fractional calculus.

In [3], the formulas are presented which give the particular solutions of those equations with inhomogeneous term in terms of the Green's function. The differential equation satisfied by the Green's function is expressed with the aid of Dirac's delta function, which is defined in distribution theory, and hence the presentation in distribution theory is adopted in [3]. In the present paper, a presentation using the AC-Laplace transform is provided. In Section 1.1, definition and preliminary formulas of the AC-Laplace transform are presented.

Let

$$p_K(t, s) := t \cdot s^2 + (c - bt)s - ab, \tag{1}$$

where $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $c \in \mathbb{C}$. Then Kummer's differential equation with an inhomogeneous term is given by

$$p_K(t, \frac{d}{dt})u(t) := t \cdot \frac{d^2}{dt^2}u(t) + (c - bt) \cdot \frac{d}{dt}u(t) - ab \cdot u(t) = f(t), \quad t > 0. \tag{2}$$

If $c \notin \mathbb{Z}$, the basic complementary solutions of (2) are given by

$$K_1(t) := {}_1F_1(a; c; bt), \tag{3}$$

$$K_2(t) := t^{1-c} \cdot {}_1F_1(a - c + 1; 2 - c; bt). \tag{4}$$

Here ${}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!(c)_k} z^k$ is the confluent hypergeometric series, $(a)_k = \prod_{l=0}^{k-1}(a + l)$ for $k \in \mathbb{Z}_{>0}$, and $(a)_0 = 1$.

Notations $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ are used to represent the sets of all real numbers, of all complex numbers and of all integers, respectively. Notations $\mathbb{R}_{>r} := \{x \in \mathbb{R} | x > r\}$, $\mathbb{R}_{\geq r} := \{x \in \mathbb{R} | x \geq r\}$ for $r \in \mathbb{R}$, ${}_+\mathbb{C} := \{z \in \mathbb{C} | \text{Re}\, z > 0\}$, $\mathbb{Z}_{>a} := \{n \in \mathbb{Z} | n > a\}$, $\mathbb{Z}_{<b} := \{n \in \mathbb{Z} | n < b\}$ and $\mathbb{Z}_{[a,b]} := \{n \in \mathbb{Z} | a \leq n \leq b\}$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ satisfying $a < b$ are also used. Heaviside's step function $H(t)$ is defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{5}$$

and when $f(t)$ is defined on $\mathbb{R}_{>\tau}$, $f(t)H(t - \tau)$ is equal to $f(t)$ for $t > \tau$ and to 0 for $t \leq \tau$. $\mathcal{L}^1_{loc}(\mathbb{R})$ is used to denote the class of functions which are locally integrable on $\mathbb{R}$.

When $c \in \mathbb{C}$ satisfies Re $(1 - c) > -1$, the solution $K_2(t)$ has the Laplace transform:

$$\hat{K}_2(s) = \mathcal{L}[K_2(t)] := \int_0^{\infty} K_2(t)e^{-st}dt, \tag{6}$$

and is obtained by solving the Laplace transform of equation (2). In [1, 2], it is confirmed that the solution $K_2(t)$ is obtained by using the AC-Laplace transform for all nonzero values of $c \in \mathbb{C} \backslash \mathbb{Z}_{>0}$.

The complementary solution of the hypergeometric differential equation, corresponding to $K_2(t)$, is found to be obtained in the form of a series of powers of s^{-1} multipied by a power of s, which has zero range of convergence. In fact, the series is the asymptotic expansion of Kummer's function $U(a, b, z)$; see Section 13.5 in [4], and is discussed also in [5]. Even in that case, by the term-by-term inverse transform, we obtain the desired result. The calculation was justified by distribution theory [2].

In [3], particular solutions of Kummer's and the hypergeometric differential equation are presented in terms of the Green's function with the aid of distribution theory. In the present paper, they are presented by using the AC-Laplace transform.

In Section 2, we give formulas of fractional calculus and the AC-Laplace transform and the norm associated to the AC-Laplace transform. We use them in giving the particular solution of differential equation with polynomial coefficients in terms of the Green's function in Section 3, and the solutions are obtained by this method for Kummer's and the hypergeometric differential equation in Sections 4 and 5, respectively, and for a fractional differential equation with coefficients of polynomial of at most first degree, in Section 6.

In [6, 7], stimulated by Yosida's works [8, 9] on Laplace's differential equations, of which typical one is Kummer's equation, the solution of Kummer's equation and a simple fractional differential equation was studied on the basis of fractional calculus and distribution theory. In [1], it was discussed in terms of the AC-Laplace transform. In [2], the arguments in [1] were applied to the solution of the homogeneous hypergeometric equation. In [3], the solution of inhomogeneous equations was discussed in terms of the Green's function and distribution theory. We now study it in terms of the Green's function and the AC-Laplace transform.

In [10, 11], the solution of inhomogeneous differential equation with constant coefficients is discussed in terms of the Green's function and distribution theory. In [3], it was discussed in terms of the Green's function and the AC-Laplace transform, where we obtain the solution which is not obtained with the aid of the usual Laplace transform.

In many sections, the same problems are taken up in [3] and the present paper, by using distribution theory and the AC-Laplace transform, respectively. In the corresponding situations, the same descriptions are adopted, e,g, in the paragraph including Equations (1)∼(4), the first two paragraphs in Section 5 in the present paper, and so on.

1.1 Preliminary formulas of the AC-Laplace transform

Definition 1.1. Let $f(a, t)$ be such a function of $t \in \mathbb{R}_{>0}$ and $a \in D_0 \subset \mathbb{C}$, that

1. $f(a, t)$ is analytic as a function of a in the domain D_0 for fixed $t \in \mathbb{R}_{>0}$,
2. the Laplace transform $\hat{f}(a, s)$ defined by

$$\hat{f}(a, s) := \mathcal{L}[f(a, t)] = \int_0^\infty f(a, t)e^{-st}dt, \tag{7}$$

   exists if $a \in D_1 \subset D_0$ and is analytic as a function of a in the domain D_1,
3. $\hat{f}(a, s)$ defined by (7) is analytic as a function of a in the domain D_0.

Then we call the analytic continuation as a function of a of $\hat{f}(a, s)$ to the domain D_0, the AC-Laplace transform of $f(a, t)$ and denote it by $\hat{f}(a, s) = \tilde{\mathcal{L}}[f(a, t)]$ for $a \in D_0$.

In solving a differential equation, the solution $u(t)$ for $t > 0$ is often assumed to be expressed as a linear combination of

$$g_\nu(t) := \frac{1}{\Gamma(\nu)}t^{\nu-1}, \quad \nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \tag{8}$$

where $\Gamma(\nu)$ is the gamma function. The Laplace transform of $g_\nu(t)$ is given by $\mathcal{L}[g_\nu(t)] = s^{-\nu}$ if $\nu \in {}_+\mathbb{C}$. We introduce the AC-Laplace transform of $g_\nu(t)$, which is expressed by $\tilde{\mathcal{L}}[g_\nu(t)]$, as in

[1, 2], such that

$$\hat{g}_\nu(s) = \tilde{\mathcal{L}}[g_\nu(t)] = s^{-\nu}, \quad \nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}. \tag{9}$$

The derivative of $g_\nu(t)$ of order $l \in \mathbb{Z}_{>0}$ is calculated by

$$\frac{d^l}{dt^l} g_\nu(t) = \begin{cases} g_{\nu-l}(t), & \nu - l \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu - l \in \mathbb{Z}_{<1}. \end{cases} \tag{10}$$

The AC-Laplace transform of $\frac{d^l}{dt^l} g_\nu(t)$ is given by

$$\tilde{\mathcal{L}}[\frac{d^l}{dt^l} g_\nu(t)] = s^{l-\nu} - \langle s^{l-\nu} \rangle_0, \tag{11}$$

where

$$\langle s^{l-\nu} \rangle_0 = \begin{cases} s^k, & k = l - \nu \in \mathbb{Z}_{>-1}, \\ 0, & l - \nu \notin \mathbb{Z}_{>-1}. \end{cases} \tag{12}$$

We note here the formulas:

$$t \cdot g_\nu(t) = t \cdot \frac{t^{\nu-1}}{\Gamma(\nu)} = \nu \cdot \frac{t^\nu}{\Gamma(\nu+1)} = \nu g_{\nu+1}(t), \tag{13}$$

$$-\frac{d}{ds} s^{-\nu} = \nu s^{-\nu-1} = \nu \tilde{\mathcal{L}}[g_{\nu+1}(t)]. \tag{14}$$

By using these, we confirm that

$$\tilde{\mathcal{L}}[t^m g_\nu(t)] = (-1)^m \frac{d^m}{ds^m} s^{-\nu}. \tag{15}$$

Condition 1.1. $u(t)$ is expressed by a linear combination of $g_\nu(t)$ for $\nu \in S$, where S is an enumerable set of $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ satisfying $\mathrm{Re}\,\nu > -M$ for some $M \in \mathbb{Z}_{>-1}$.

Remark 1.1. The complementary solutions of a differential equation with polynomial coefficients usually satisfy this condition.

When $u(t)$ satisfies Condition 1.1, it is expressed as follows:

$$u(t) = \sum_{\nu \in S} u_{\nu-1} g_\nu(t) = \sum_{\nu \in S} u_{\nu-1} \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{16}$$

where $u_{\nu-1} \in \mathbb{C}$ are constants. When $\hat{u}(s) = \tilde{\mathcal{L}}[u(t)]$ exists, it is expressed by

$$\hat{u}(s) = \sum_{\nu \in S} u_{\nu-1} \hat{g}_\nu(s) = \sum_{\nu \in S} u_{\nu-1} s^{-\nu}. \tag{17}$$

By applying formulas (15) and (11), we obtain

Lemma 1.1. *Let $m \in \mathbb{Z}_{>0}$, $l \in \mathbb{Z}_{>0}$, $u(t)$ be expressed by* (16) *and $\hat{u}(s) := \tilde{\mathcal{L}}[u(t)]$. Then*

$$\tilde{\mathcal{L}}[t^m u(t)] = (-1)^m \frac{d^m}{ds^m} \hat{u}(s), \tag{18}$$

$$\tilde{\mathcal{L}}[\frac{d^l}{dt^l} u(t)] = s^l \hat{u}(s) - \langle s^l \hat{u}(s) \rangle_0, \tag{19}$$

$$\tilde{\mathcal{L}}[t^m \frac{d^l}{dt^l} u(t)] = (-1)^m \frac{d^m}{ds^m} [s^l \hat{u}(s)] - (-1)^m \frac{d^m}{ds^m} \langle s^l \hat{u}(s) \rangle_0, \tag{20}$$

where

$$\langle s^l \hat{u}(s)\rangle_0 = \sum_{k=0}^{l-1} u_{l-k-1} s^k. \tag{21}$$

In particular,

$$\langle s\hat{u}(s)\rangle_0 = u_0, \quad \langle s^2\hat{u}(s)\rangle_0 = u_0 s + u_1, \quad \langle s^3\hat{u}(s)\rangle_0 = u_0 s^2 + u_1 s + u_2. \tag{22}$$

Remark 1.2. Even when the series on the righthand side of (17) does not converge for any s, the series on the righthand side of (16) may converge in an interval of t on $\mathbb{R}$. In such a case, we use $\hat{u}(s) = \mathcal{L}_S[u(t)]$ to represent the series on the righthand side of (17). Then the operations on $\hat{u}(s)$ are supposed to be done term-by-term [2].

2 Fractional Derivative and the AC-Laplace Transform

We consider the Riemann-Liouville fractional integral and derivatives ${}_cD_R^\mu f(z)$ of order $\mu \in \mathbb{C}$, when we may usually discuss the derivative $\frac{d^n}{dz^n} f(z) = f^{(n)}(z)$ of order $n \in \mathbb{Z}_{>0}$: see [12] and Section 2.3.2 in [13]. In the following definition, $P(c, z)$ is the path from $c \in \mathbb{C}$ to $z \in \mathbb{C}$, and $\mathcal{L}^1(P(c, z))$ is the class of functions which are integrable on $P(c, z)$, and $\lfloor x \rfloor$ for $x \in \mathbb{R}$ denotes the greatest integer not exceeding x.

Definition 2.1. Let $c \in \mathbb{C}$, $z \in \mathbb{C}$, $f(\zeta) \in \mathcal{L}^1(P(c, z))$, and $f(\zeta)$ be continuous in a neighborhood of $\zeta = z$. Then the Riemann-Liouville fractional integral of order $\lambda \in {}_+\mathbb{C}$ is defined by

$${}_cD_R^{-\lambda} f(z) ::= \frac{1}{\Gamma(\lambda)} \int_c^z (z-\zeta)^{\lambda-1} f(\zeta)\, d\zeta, \tag{23}$$

and the Riemann-Liouville fractional derivative of order $\mu \in \mathbb{C}$ satisfying $\mathrm{Re}\,\mu \geq 0$ is defined by

$${}_cD_R^\mu f(z) := {}_cD_R^l[{}_cD_R^{\mu-l} f(z)], \tag{24}$$

when the righthand side exists, where $l = \lfloor \mathrm{Re}\,\mu \rfloor + 1$, and ${}_cD_R^l f(z) = \frac{d^l}{dz^l} f(z) = f^{(l)}(z)$ for $l \in \mathbb{Z}_{>-1}$.

In the following study, the value $c = 0$ is chosen.

In place of (10), (11), (19) and (20), the following lemmas hold valid [1, 2].

Lemma 2.1. *Let $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $\mu \in \mathbb{C}$. Then for $t \in \mathbb{R}_{>0}$, we have*

$${}_0D_R^\mu g_\nu(t) = \begin{cases} g_{\nu-\mu}(t), & \nu-\mu \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu-\mu \in \mathbb{Z}_{<1}. \end{cases} \tag{25}$$

Lemma 2.2. *Let $g_\nu(t)$ and $\hat{g}_\nu(s)$ be given by (8) and (9). Then for $\mu \in \mathbb{C}$, we have*

$$\tilde{\mathcal{L}}[{}_0D_R^\mu g_\nu(t)] = s^{\mu-\nu} - \langle s^\mu \hat{g}_\nu(s)\rangle_0, \tag{26}$$

where

$$\langle s^\mu \hat{g}_\nu(s)\rangle_0 = \begin{cases} s^k, & k = \mu-\nu \in \mathbb{Z}_{>-1}, \\ 0, & \mu-\nu \notin \mathbb{Z}_{>-1}. \end{cases} \tag{27}$$

Lemma 2.3. *Let $m \in \mathbb{Z}_{>0}$, $\mu \in \mathbb{C}$, $u(t)$ be expressed by* (16), *and $\hat{u}(s) := \tilde{\mathcal{L}}[u(t)]$. Then*

$$s^{\mu}\hat{u}(s) = \tilde{\mathcal{L}}[{}_0D_R^{\mu}u(t)] + \langle s^{\mu}\hat{u}(s)\rangle_0, \tag{28}$$

$$(-1)^m \frac{d^m}{ds^m}[s^{\mu}\hat{u}(s)] = \tilde{\mathcal{L}}[t^m {}_0D_R^{\mu}u(t)] + (-1)^m \frac{d^m}{ds^m}\langle s^{\mu}\hat{u}(s)\rangle_0, \tag{29}$$

where

$$\langle s^{\mu}\hat{u}(s)\rangle_0 = \sum_{k=0,\ \mu-k\in S}^{\infty} u_{\mu-k-1}s^k. \tag{30}$$

Definition 2.2. Let $\tau \in \mathbb{R}$ and $\psi(t)H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$. Then $\mathcal{L}_\tau[\psi(t)]$ is defined by

$$\mathcal{L}_\tau[\psi(t)] := \int_0^{\infty} \psi(\tau+x)e^{-sx}dx = \int_\tau^{\infty} \psi(t)e^{-s(t-\tau)}dt. \tag{31}$$

Lemma 2.4. *Let $\tau \in \mathbb{R}$, $l \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{>0}$, $\frac{d^l}{dt^l}\psi(t) \cdot H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$, and $\hat{\psi}_\tau(s) := \mathcal{L}_\tau[\psi(t)]$. Then*

$$\mathcal{L}_\tau[t^m\psi(t)] = (\tau - \frac{d}{ds})^m\hat{\psi}_\tau(s), \tag{32}$$

$$s^l\hat{\psi}_\tau(s) = \mathcal{L}_\tau[\frac{d^l}{dt^l}\psi(t)] + \langle s^l\hat{\psi}_\tau(s)\rangle_0, \tag{33}$$

$$(\tau - \frac{d}{ds})^m[s^l\hat{\psi}_\tau(s)] = \mathcal{L}_\tau[t^m\frac{d^l}{dt^l}\psi(t)] + (\tau - \frac{d}{ds})^m\langle s^l\hat{\psi}_\tau(s)\rangle_0, \tag{34}$$

where

$$\langle s^l\hat{\psi}_\tau(s)\rangle_0 = \sum_{k=0}^{l-1}\psi^{(k)}(\tau)s^{l-k-1}. \tag{35}$$

When $l = 1$, 2 and 3, we have

$$\langle s\hat{\psi}_\tau(s)\rangle_0 = \psi(\tau), \quad \langle s^2\hat{\psi}_\tau(s)\rangle_0 = \psi(\tau)s + \psi'(\tau), \quad \langle s^3\hat{\psi}_\tau(s)\rangle_0 = \psi(\tau)s^2 + \psi'(\tau)s + \psi''(\tau). \tag{36}$$

Proof. Equation (32) is confirmed by

$$\mathcal{L}_\tau[t^m\psi(t)] = \int_\tau^{\infty} t^m\psi(t)e^{-s(t-\tau)}dt = (\tau - \frac{d}{ds})^m\hat{\psi}_\tau(s). \tag{37}$$

By integration by parts, we obtain

$$\mathcal{L}_\tau[\frac{d^l}{dt^l}\psi(t)] = \int_\tau^{\infty} e^{-s(t-\tau)}\frac{d^l}{dt^l}\psi(t)dt = -\psi^{(l-1)}(\tau) + s\cdot\mathcal{L}_\tau[\frac{d^{l-1}}{dt^{l-1}}\psi(t)], \tag{38}$$

which is used to prove (33). Equation (34) is obtained, by replacing $\psi(t)$ and $\hat{\psi}_\tau(s)$ by $\frac{d^l}{dt^l}\psi(t)$ and $\tilde{\mathcal{L}}[\frac{d^l}{dt^l}\psi(t)]$, respectively, in (32), with the aid of (33). ■

Lemma 2.5. *Let the condition in Lemma 2.4 be satisfied, and $\psi_\tau(t) = \psi(t)H(t-\tau)$. Then in place of $\mathcal{L}_\tau[\psi_\tau(t)] = \hat{\psi}(s)$ and* (32), *we have*

$$\mathcal{L}[\psi_\tau(t)] = \mathcal{L}_\tau[\psi_\tau(t)]e^{-\tau s} = \hat{\psi}_\tau(s)e^{-\tau s}, \tag{39}$$

$$\mathcal{L}[t^m\psi_\tau(t)] = (-1)^m\frac{d^m}{ds^m}\mathcal{L}[\psi_\tau(t)] = (-1)^m\frac{d^m}{ds^m}[\hat{\psi}_\tau(s)e^{-\tau s}], \tag{40}$$

$$\mathcal{L}[t^m\psi_\tau(t)] = \mathcal{L}_\tau[t^m\psi_\tau(t)]\cdot e^{-\tau s} = (\tau - \frac{d}{ds})^m\hat{\psi}_\tau(s)\cdot e^{-\tau s}. \tag{41}$$

We also have

$$(-1)^m \frac{d^m}{ds^m}[s^l \hat{\psi}_\tau(s) e^{-\tau s}] = (\tau - \frac{d}{ds})^m [s^l \hat{\psi}_\tau(s)] \cdot e^{-\tau s}. \tag{42}$$

Proof. Equation (41) is obtained with the aid of (39) and (32). Equation (42) is obtained by using

$$-\frac{d}{ds}[s^l \hat{\psi}_\tau(s) e^{-\tau s}] = (\tau - \frac{d}{ds})[s^l \hat{\psi}_\tau(s)] \cdot e^{-\tau s}, \tag{43}$$

repeatedly. ■

3 Green's Function for Inhomogeneous Differential Equations with Polynomial Coefficients

Let

$$p_n(t, s) := \sum_{l=0}^{n} a_l(t) s^l = \sum_{l=0}^{n} \sum_{m=0}^{2} a_{l,m} t^m s^l, \tag{44}$$

where $n \in \mathbb{Z}_{>0}$, $a_{l,m} \in \mathbb{C}$ are contants, and $a_l(t) = \sum_{m=0}^{2} a_{l,m} t^m$ are polynomials of t satisfying $a_0(t) \neq 0$ and $a_n(t) \neq 0$. Discussions are made of the differential equation with an inhomogeneous term which is given by

$$p_n(t, \frac{d}{dt}) u(t) := \sum_{l=0}^{n} a_l(t) \frac{d^l}{dt^l} u(t) = \sum_{l=0}^{n} \sum_{m=0}^{2} a_{l,m} t^m \frac{d^l}{dt^l} u(t) = f(t), \quad t > 0. \tag{45}$$

Remark 3.1. In Section 4, we consider Kummer's differential equation given by (2), that is Equation (45) in which

$$n = 2, \quad a_2(t) = t, \quad a_1(t) = c - bt, \quad a_0(t) = -ab. \tag{46}$$

In Section 5, we consider the hypergeometric differential equation, that is Equation (45) in which

$$n = 2, \quad a_2(t) = t(1-t), \quad a_1(t) = c - (a+b+1)t, \quad a_0(t) = -ab. \tag{47}$$

These are special ones of

$$n = 2, \quad a_2(t) = t + a_{2,2} t^2, \quad a_1(t) = c + a_{1,1} t, \quad a_0(t) = a_{0,0}. \tag{48}$$

For the inhomogeneous term $f(t)$, we consider the following three cases.

Condition 3.1. (i) $f(t)H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$,

(ii) $f(t) = {}_0D_R^\beta f_\beta(t)$, where $f_\beta(t)H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$, and $\hat{f}(s) = s^\beta \hat{f}_\beta(s)$,

(iii) $f(t) = g_{-\beta}(t) = \frac{1}{\Gamma(-\beta)} t^{-\beta-1}$, $\hat{f}(s) = \hat{g}_{-\beta}(s) = s^\beta$ and $\beta \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$.

Lemma 3.1. *Let $u(t)$ be expressed by (16) and be a solution of (45), Then the differential equation satisfied by $\hat{u}(s) := \tilde{\mathcal{L}}[u(t)]$ is*

$$\begin{aligned} p_n(-\frac{d}{ds}, s)\hat{u}(s) &:= \sum_{l=0}^{n} a_l(-\frac{d}{ds})[s^l \hat{u}(s)] = \tilde{\mathcal{L}}[p_n(t, \frac{d}{dt})u(t)] + \sum_{l=1}^{n} a_l(-\frac{d}{ds})\langle s^l \hat{u}(s)\rangle_0 \\ &= \hat{f}(s) + \sum_{l=1}^{n} a_l(-\frac{d}{ds})\langle s^l \hat{u}(s)\rangle_0, \end{aligned} \tag{49}$$

where $\hat{f}(s) = \tilde{\mathcal{L}}[f(t)]$, *and* $\langle s^l \hat{u}(s)\rangle_0$ *are given by* (21) *and* (22). *When* (48) *applies, the second term in the righthand side of Equation* (49) *is expressed by*

$$\sum_{l=1}^{2} a_l(-\frac{d}{ds})\langle s^l \hat{u}(s)\rangle_0 = (c-1)u_0. \tag{50}$$

Proof. Equation (49) is obtained by using (20). Equation (50) is confirmed by using (48) and (22) in the lefthand side of (50). ■

Lemma 3.2. *Let* $\tau \in \mathbb{R}$, *Equation* (45) *be satisfied by* $u(t) = \psi(t)$ *for* $t > \tau$, $\frac{d^n}{dt^n}\psi(t) \cdot H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$, *and* $\hat{\psi}_\tau(s) := \mathcal{L}_\tau[\psi(t)]$. *Then*

$$p_n(\tau - \frac{d}{ds}, s)\hat{\psi}_\tau(s) := \sum_{l=0}^{n} a_l(\tau - \frac{d}{ds})[s^l \hat{\psi}_\tau(s)] = \mathcal{L}_\tau[p_n(t, \frac{d}{dt})\psi(t)] + \sum_{l=1}^{n} a_l(\tau - \frac{d}{ds})\langle s^l \hat{\psi}_\tau(s)\rangle_0, \tag{51}$$

where $\langle s^l \hat{\psi}_\tau(s)\rangle_0$ *is given by* (35) *and* (36). *When* (48) *applies, the second term in the righthand side of Equation* (51) *is expressed by*

$$\sum_{l=1}^{2} a_l(\tau - \frac{d}{ds})\langle s^l \hat{\psi}_\tau(s)\rangle_0 = (\tau + a_{2,2}\tau^2)\psi'(\tau) + (\tau s - 1 - 2a_{2,2}\tau + c + a_{1,1}\tau)\psi(\tau). \tag{52}$$

Proof. Equation (51) is obtained with the aid of (34). Equation (52) is confirmed, by using (48) and (36) in the lefthand side of (52). ■

Let Condition 3.1(i) be satisfied. Then $\hat{f}(s) = \mathcal{L}[f(t)]$.

Definition 3.1. For Equation (45), the Green's function $G(t,\tau)$ for fixed $\tau \in \mathbb{R}_{\geq 0}$ is such that $\hat{G}(s,\tau) := \mathcal{L}_\tau[G(t,\tau)]$ satisfies

$$p_n(\tau - \frac{d}{ds}, s)\hat{G}(s,\tau) = \sum_{l=0}^{n} a_l(\tau - \frac{d}{ds})[s^l \hat{G}(s,\tau)] = 1. \tag{53}$$

Lemma 3.3. *The Green's function* $G(t,\tau)$ *for Equation* (45) *satisfies*

$$p_n(t, \frac{d}{dt})G(t,\tau) = \sum_{l=0}^{n} a_l(t)\frac{d^l}{dt^l}G(t,\tau) = 0, \quad t > \tau, \tag{54}$$

and $G(t,\tau) = 0$ *for* $t < \tau$. *The values of* $G(t,\tau)$ *and its derivatives at* $t = \tau$ *are determined by*

$$\sum_{l=1}^{n} a_l(\tau - \frac{d}{ds})\langle s^l \hat{G}(s,\tau)\rangle_0 = 1, \tag{55}$$

where $\langle s^l \hat{G}(s,\tau)\rangle_0$ *is given by* (35) *and* (36) *with* $\hat{\psi}_\tau(s)$ *and* $\psi(\tau)$ *replaced by* $\hat{G}(s,\tau)$ *and* $G(\tau,\tau)$, *respectively.*

Proof. This is confirmed by comparing (53) with (51), where $\psi(t)$ and $\hat{\psi}_\tau(s)$ are replaced by $G(t,\tau)$ and $\hat{G}(s,\tau)$, respectively. ■

Lemma 3.4. *Let* (48) *apply,* $u(t)$ *satisfy* $p_2(t, \frac{d}{dt})u(t) = 0$ *for* $t > 0$, *and* $u(0) = u_0$. *Then the Green's function* $G(t,0)$ *for Equation* (45) *is given by*

$$G(t,0) = \frac{1}{(c-1)u_0}u(t)H(t). \tag{56}$$

Proof. In this case, (50) shows that the righthand side of Equation (49) is equal to $(c-1)u_0$. By comparing this with (55) for $\tau = 0$, we conclude the proof. ∎

Lemma 3.5. *Let* (48) *apply,* $\tau > 0$, $\psi(t)$ *satisfy* $p_2(t, \frac{d}{dt})\psi(t) = 0$ *for* $t > \tau$, *and* $\psi(\tau) = 0$. *Then the Green's function* $G(t,\tau)$ *for Equation* (45) *is given by*

$$G(t,\tau) = \frac{1}{a_2(\tau)\psi'(\tau)}\psi(t)H(t-\tau). \tag{57}$$

Proof. In this case, (52) shows that the righthand side of Equation (51) is equal to $(\tau + a_{2,2}\tau^2)\psi'(\tau) = a_2(\tau)\psi'(\tau)$. By comparing this with (55), we conclude the proof. ∎

Lemma 3.6. *Let* $\hat{G}(s,\tau) = \mathcal{L}_\tau[G(t,\tau)]$ *as in Definition 3.1. Then*

$$\mathcal{L}[G(t,\tau)] = \hat{G}(s,\tau)e^{\tau s}. \tag{58}$$

Proof. This is due to formula (39). ∎

Lemma 3.7. *The following equation is equivalent to* (53) *in Definition 3.1:*

$$p_n(-\frac{d}{ds}, s)[\hat{G}(s,\tau)e^{-\tau s}] = \sum_{l=0}^{n} a_l(-\frac{d}{ds})[s^l \hat{G}(s,\tau)e^{-\tau s}] = e^{-\tau s}. \tag{59}$$

Proof. This is confirmed by using formula (42). ∎

Lemma 3.8. *Let* $G(t,\tau)$ *be defined by Definition 3.1 for Equation* (45), *Condition 3.1(i) be satisfied, and* $u_f(t)$ *be given by*

$$u_f(t) := \int_0^t G(t,\tau)f(\tau)d\tau. \tag{60}$$

Then

$$\hat{u}_f(s) := \mathcal{L}[u_f(t)] = \int_0^\infty \hat{G}(s,\tau)f(\tau)e^{-s\tau}d\tau, \tag{61}$$

is a particular solution of (49) *for the term* $\hat{f}(s)$, *and*

$$\mathcal{L}[p_n(t, \frac{d}{dt})u_f(t)] = \hat{f}(s). \tag{62}$$

Proof. Equation (61) is due to Lemma 3.6. We confirm the rest of the statements by using formula (59) in (49) for $\hat{u}(s) = \hat{u}_f(s)$. ∎

This lemma implies.

Corollary 3.1. $u_f(t)$ and $\hat{u}_f(s)$ given by (60) and (61) are particular solutions of (45) and (49) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Lemma 3.9. *Let* (48) *apply,* $\hat{u}(s)$ *be a solution of* (49) *with* (50), $\tilde{p}_2(t,s)$ *be related with* $p_2(t,s)$ *given by* (44) *for* $n = 2$, *by*

$$\tilde{p}_2(t,s) := s^{-\beta}p_2(t,s)s^\beta = s^{-\beta}[(t + a_{2,2}t^2)\cdot s^2 + (c + a_{1,1}t)\cdot s + a_{0,0}]s^\beta, \tag{63}$$

and $\hat{f}_\beta(s) = s^{-\beta}\hat{f}(s)$*, where* $t = -\frac{d}{ds}$*. Then*

$$\tilde{p}_2(t,s) = (t + a_{2,2}t^2)\cdot s^2 + (c - \beta + (a_{1,1} - 2\beta a_{2,2})t)\cdot s + a_{2,2}\beta(\beta+1) - a_{1,1}\beta + a_{0,0}, \quad (64)$$

and $\hat{w}(s) = s^{-\beta}\hat{u}(s)$ *satisfies*

$$\tilde{p}_2(-\frac{d}{ds}, s)\hat{w}(s) = \hat{f}_\beta(s) + u_0(c-1)s^{-\beta}. \quad (65)$$

Proof. (64) is obtained from (63) with the aid of the following lemma. ■

Lemma 3.10. *Let* $\hat{v}(s) = \tilde{\mathcal{L}}[v(t)]$*. Then*

$$\frac{d}{ds}[s^\beta\cdot s\hat{v}(s)] = s^\beta\{\beta\hat{v}(s) + \frac{d}{ds}[s\hat{v}(s)]\}, \quad (66)$$

$$\frac{d^2}{ds^2}[s^\beta\cdot s^2\hat{v}(s)] = s^\beta\{\beta^2\hat{v}(s) + \beta\frac{d}{ds}[s\hat{v}(s)] + \beta s^{-1}\frac{d}{ds}[s^2\hat{v}(s)] + \frac{d^2}{ds^2}[s^2\hat{v}(s)]\}$$
$$= s^\beta\{\beta(\beta+1)\hat{v}(s) + 2\beta\frac{d}{ds}[s\hat{v}(s)] + \frac{d^2}{ds^2}[s^2\hat{v}(s)]\}. \quad (67)$$

Corollary 3.1 shows that the particular solution of (65) for the term $\hat{f}_\beta(s) = \tilde{\mathcal{L}}[f_\beta(t)]$ is expressed by a particular solution of

$$p_\beta(t, \frac{d}{dt})w(t) = f_\beta(t), \quad t > 0. \quad (68)$$

4 Particular Solution of Kummer's Differential Equation in Terms of the Green's Function

Kummer's differential equation with an inhomogeneous term $f(t)$ is given in (2). If $f(t) = 0$ and $c \notin \mathbb{Z}$, the basic solutions $K_1(t)$ and $K_2(t)$ of (2) are given by (3) and (4).

We now obtain a particular solution of this equation by the method stated in Section 3.

4.1 Solution of Equation (2) in which Condition 3.1(i) is satisfied

The following lemma is a special one of Lemma 3.1 for the case where (46) applies.

Lemma 4.1. *Let* $u(t)$ *be expressed by* (16) *and be a solution of Equation* (2)*. Then the differential equation satisfied by* $\hat{u}(s) = \tilde{\mathcal{L}}[u(t)]$ *is given by*

$$p_K(-\frac{d}{ds}, s)\hat{u}(s) = \mathcal{L}_H[p_K(t, \frac{d}{dt})u(t)] + u_0(c-1) = \hat{f}(s) + u_0(c-1). \quad (69)$$

The following lemma is a special one of Lemma 3.2 for the case where (46) applies.

Lemma 4.2. *Let* $\tau \geq 0$*,* $p_K(t, \frac{d}{dt})\psi(t)\cdot H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$ *and* $\hat{\psi}_\tau(s) = \mathcal{L}_\tau[\psi(t)]$*. Then*

$$p_K(\tau - \frac{d}{ds}, s)\hat{\psi}_\tau(s) = \mathcal{L}_\tau[p_K(t, \frac{d}{dt})\psi(t)] + \tau\psi'(\tau) + \psi(\tau)(c - b\tau - 1) + \psi(\tau)\tau s. \quad (70)$$

The following lemma is a special one of Lemmas 3.4 and 3.5 for the case where (46) applies.

Lemma 4.3. *Let $K_1(t)$ and $K_2(t)$ be given by (3) and (4), and $\psi_K(t,\tau)$ for fixed $\tau > 0$ be given by*

$$\psi_K(t,\tau) := c_1(\tau) \cdot K_1(t) + c_2(\tau) \cdot K_2(t), \tag{71}$$

where $c_1(\tau)$ and $c_2(\tau)$ are constants which depend on τ and are so chosen that $\psi_K(\tau,\tau) = 0$. Then $G_K(t,0)$ and $G_K(t,\tau)$ for $\tau > 0$, which are given by

$$G_K(t,0) := \frac{1}{c-1} \cdot K_1(t)H(t), \quad G_K(t,\tau) := \frac{1}{\tau\psi'_K(\tau,\tau)}\psi_K(t,\tau)H(t-\tau), \tag{72}$$

are the Green's functions for Equation (2), so that $\hat{G}_K(s,0) = \mathcal{L}[G_K(t,0)]$ and $\hat{G}_K(s,\tau) = \mathcal{L}_\tau[G_K(t,\tau)]$ satisfy

$$p_K(\tau - \frac{d}{ds}, s)\hat{G}_K(s,\tau) = 1, \tag{73}$$

for $\tau = 0$ and $\tau > 0$, respectively.

Proof. If we put $u(t) = \frac{1}{c-1} \cdot K_1(t)$ in (69), then we have $\hat{u}(s) = \hat{G}_K(s,0)$ on the lefthand side of (69), and the righthand side of (69) is $u(0)(c-1) = 1$, since $p_K(t, \frac{d}{dt})K_1(t) = 0$ and $K_1(0) = 1$. Thus (69) guarantees (73) for $\tau = 0$. If we put $\psi(t) = \frac{1}{\tau\psi'_K(\tau,\tau)}\psi_K(t,\tau)$ and $\psi(\tau) = 0$ in (70), then we have $\hat{\psi}_\tau(s) = \hat{G}_K(s,\tau)$ on the lefthand side of (70), and the righthand side of (70) is $\tau \cdot \psi'(\tau) = 1$. Thus (70) guarantees (73) for $\tau > 0$. ■

Theorem 4.1. Let $G_K(t,\tau)$ and $\psi_K(t,\tau)$ be those given in Lemma 4.3, Condition 3.1(i) be satisfied, and $u_f(t)$ be given by

$$u_f(t) := \int_0^t G_K(t,\tau)f(\tau)d\tau = \int_0^t \psi_K(t,\tau)\frac{f(\tau)}{\tau\psi'_K(\tau,\tau)}d\tau. \tag{74}$$

Then

$$\hat{u}_f(s) = \int_0^\infty \hat{G}_K(s,\tau)f(\tau)e^{-s\tau}d\tau = \int_0^\infty \hat{\psi}_K(s,\tau)\frac{f(\tau)}{\tau\psi'_K(\tau,\tau)}e^{-s\tau}d\tau. \tag{75}$$

Now $u_f(t)$ and $\hat{u}_f(t)$ are particular solutions of (2) and (69) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Proof. This is guaranteed by Corollary 3.1. ■

Remark 4.1. By using the first equation in (73), we see that the particular solution of (69) for the last term, is

$$\hat{u}_1(s) = u_0(c-1)\hat{G}_K(s,0). \tag{76}$$

The corresponding complementary solution of (2) is

$$u_1(t) = u_0(c-1)G_K(t,0) = u_0 \cdot K_1(t). \tag{77}$$

Considering that the basic complementary solutions of (2) are given by (3) and (4), the general solution of (2) is now given by

$$u(t) = u_f(t) + u_0 \cdot K_1(t) + u_{1-c} \cdot K_2(t). \tag{78}$$

The condition $\psi_K(\tau,\tau) = 0$ requires that $c_1(\tau) \cdot K_1(\tau) = -c_2(\tau) \cdot K_2(\tau)$, and hence we may choose $\psi_K(t,\tau)$ as

$$\psi_K(t,\tau) = \begin{cases} K_1(t) - \frac{K_1(\tau)}{K_2(\tau)} \cdot K_2(t), & |K_1(\tau)| < |K_2(\tau)|, \\ \frac{K_2(\tau)}{K_1(\tau)} \cdot K_1(t) - K_2(t), & |K_1(\tau)| \geq |K_2(\tau)|. \end{cases} \tag{79}$$

Remark 4.2. In [6, 7, 1, 2], the Laplace transform of $K_2(t)$ was obtained by solving the first order differential equation (69) for $\hat{f}(s) = 0$ and $u_0 = 0$. In [6, 7, 1], the Laplace transform of $K_1(t)$ was obtained by solving the same equation for $\hat{f}(s) = 0$ and $u_0 = 1$.

4.2 Solution of Equation (2) in which Condition 3.1(ii) is satisfied

We give the solution of equation (2) of which the inhomogeneous term $f(t)$ satisfies Condition 3.1(ii), so that $\hat{f}(s) = s^\beta \hat{f}_\beta(s)$, and $f_\beta(t)$ satisfies $f_\beta(t)H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$.

The following lemma is a special one of Lemma 3.9 for the case where (46) applies.

Lemma 4.4. *Let $\hat{u}(s)$ be a solution of* (69)*, and $\tilde{p}_K(-\frac{d}{ds}, s)$ be related with $p_K(-\frac{d}{ds}, s)$ given by* (1)*, by*

$$\tilde{p}_K(-\frac{d}{ds}, s) := s^{-\beta} p_K(-\frac{d}{ds}, s) s^\beta. \tag{80}$$

Then

$$\tilde{p}_K(t, s) := t \cdot s^2 + (c - \beta - bt)s - (a - \beta)b, \tag{81}$$

and $\hat{w}(s) = s^{-\beta}\hat{u}(s)$ satisfies

$$\tilde{p}_K(-\frac{d}{ds}, s)\hat{w}(s) = \hat{f}_\beta(s) + u_0(c-1)s^{-\beta}. \tag{82}$$

Theorem 4.1 shows that the particular solution of (82) for the term $\hat{f}_\beta(s)$ is expressed by a particular solution of

$$\tilde{p}_K(t, \frac{d}{dt})w(t) = f_\beta(t), \quad t > 0. \tag{83}$$

Remark 4.3. We note that $\tilde{p}_K(t, s)$ given by (81) is obtained from $p_K(t, s)$ given by (1), by replacing a and c by $a - \beta$ and $c - \beta$, respectively, and hence the complementary solutions $K_{\beta,1}(t)$ and $K_{\beta,2}(t)$ and Green's functions $G_{\tilde{K}}(t, 0)$ and $G_{\tilde{K}}(t, \tau)$ of Equation (83) are obtained from those $K_1(t)$, $K_2(t)$, $G_K(t, 0)$ and $G_K(t, \tau)$, respectively, of Equation (2), by the same replacement.

Now in place of Theorem 4.1, we have the following theorem.

Theorem 4.2. Let $G_{\tilde{K}}(t, \tau)$ and $\psi_{\tilde{K}}(t, \tau)$ be obtained from $G_K(t, \tau)$ and $\psi_K(t, \tau)$ by the replacement stated in Remark 4.3, Condition 3.1(ii) be satisfied for $\beta \notin \mathbb{Z}_{>0}$, and $w_g(t)$ be given by

$$w_g(t) := \int_0^t G_{\tilde{K}}(t, \tau) f_\beta(\tau) d\tau = \int_0^t \psi_{\tilde{K}}(t, \tau) \frac{f_\beta(\tau)}{\tau \psi'_{\tilde{K}}(\tau, \tau)} d\tau. \tag{84}$$

Then

$$\hat{w}_g(s) = \int_0^\infty \hat{G}_{\tilde{K}}(s, \tau) f_\beta(\tau) e^{-s\tau} d\tau = \int_0^\infty \hat{\psi}_{\tilde{K}}(s, \tau) \frac{f_\beta(\tau)}{\tau \psi'_{\tilde{K}}(\tau, \tau)} e^{-s\tau} d\tau. \tag{85}$$

Now $u_f(t) := {}_0D^\beta_R w_g(t)$ and $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$ are particular solutions of (2) and (69) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Proof. Theorem 4.1 states that when $w_g(t)$ is given by (84), $\hat{w}_g(s) = \mathcal{L}[w_g(t)]$ is the particular solution of (82) for the term $\hat{f}_\beta(s)$, and Lemma 4.4 states that $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$ is the particular solution of (69) for the term $\hat{f}(s)$. With the aid of Equation (19) in Lemma 2.3, we confirm that if $u_f(t)$ is given by $u_f(t) = {}_0D^\beta_R w_g(t)$, then $\hat{u}_f(s) = \tilde{\mathcal{L}}[u_f(t)]$ is given by $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$, when $\beta \notin \mathbb{Z}_{>0}$. ■

4.3 Solution of Equation (2) in which Condition 3.1(iii) is satisfied

We give the solution of Equation (2) of which the inhomogeneous term $f(t)$ satisfies Condition 3.1(iii), so that $f(t) = g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ and $\hat{f}(s) = \hat{g}_\nu(s) = s^{-\nu}$. Here we use $\beta \notin \mathbb{Z}_{>-1}$ in place of $-\nu$.

Lemma 4.5. *Let $\hat{f}(s) = s^\beta$, $\tilde{p}_K(t,s)$ be given by (81), and $\hat{G}_{\tilde{K}}(s,0)$ satisfy*

$$\tilde{p}_K(t,s)\hat{G}_{\tilde{K}}(s,0) = 1. \tag{86}$$

Then the particular solution of (69) for the term $\hat{f}(s) = s^\beta$ is given by

$$\hat{u}_f(s) = s^\beta \hat{G}_{\tilde{K}}(s,0). \tag{87}$$

Proof. $\hat{u}_f(s)$ satisfies $p_K(t,s)\hat{u}_f(s) = s^\beta$ and hence $\tilde{p}_K(t,s)s^{-\beta}\hat{u}_f(s) = 1$ by (80). Comparing this with (86), we see that (87) is satisfied. ■

By Remark 4.3, with the aid of $G_K(t,0)$ given by (72) and (3), we have

Lemma 4.6. *As the solution of (86), we obtain*

$$\hat{G}_{\tilde{K}}(s,0) = \mathcal{L}[G_{\tilde{K}}(t,0)], \quad G_{\tilde{K}}(t,0) = \frac{1}{c-\beta-1} \cdot {}_1F_1(a-\beta; c-\beta; bt). \tag{88}$$

Theorem 4.3. Let Condition 3.1(iii) be satisfied, and $G_{\tilde{K}}(t,0)$ be given by (88). Then the particular solution of (2) is given by

$$u_f(t) := {}_0D_R^\beta G_{\tilde{K}}(t,0) = \frac{t^{-\beta}}{(c-\beta-1)\Gamma(1-\beta)} \cdot {}_2F_2(1, a-\beta; 1-\beta, c-\beta; bt), \tag{89}$$

where ${}_2F_2(a_1,a_2;c_1,c_2;z) = \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k}{k!(c_1)_k(c_2)_k} z^k$.

Proof. We confirm that when $u_f(t)$ is given by (89), $\hat{u}_f(s) = \tilde{\mathcal{L}}[u_f(t)]$ is given by (87) and (88), with the aid of Equation (19) in Lemma 2.3. ■

5 Particular Solution of the Hypergeometric Differential Equation in Terms of the Green's Function

Let

$$p_H(t,s) := t(1-t)\cdot s^2 + (c-(a+b+1)t)\cdot s - ab, \tag{90}$$

where $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $c \in \mathbb{C}$ are constants. Then the hypergeometric differential equation with an inhomogeneous term $f(t)$ is given by

$$p_H(t, \frac{d}{dt})u(t) := t(1-t)\cdot \frac{d^2}{dt^2}u(t) + (c-(a+b+1)t)\cdot \frac{d}{dt}u(t) - ab\cdot u(t) = f(t), \quad t > 0. \tag{91}$$

If $f(t) = 0$ and $c \notin \mathbb{Z}$, the basic solutions of (91) in [4] and [14] are given by

$$H_1(t) := {}_2F_1(a,b;c;t), \tag{92}$$

$$H_2(t) := t^{1-c} \cdot {}_2F_1(1+a-c, 1+b-c; 2-c; t), \tag{93}$$

where ${}_2F_1(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{k!(c)_k} z^k$ of $z \in \mathbb{C}$ is the hypergeometric series.

We now obtain a particular solution of this equation by the method stated in Section 3 and used in Section 4.1.

5.1 Solution of Equation (91) in which Condition 3.1(i) is satisfied

The following lemma is a special one of Lemma 3.1 for the case where (47) applies.

Lemma 5.1. *Let $u(t)$ be expressed by* (16) *and be a solution of Equation* (91)*. Then the differential equation satisfied by $\hat{u}(s) = \mathcal{L}_S[u(t)]$ is given by*

$$p_H(-\frac{d}{ds}, s)\hat{u}(s) = \mathcal{L}_H[p_H(t, \frac{d}{dt})u(t)] + u_0(c-1) = \hat{f}(s) + u_0(c-1). \tag{94}$$

The following lemma is a special one of Lemma 3.2 for the case where (47) applies.

Lemma 5.2. *Let $\tau \geq 0$, $p_H(t, \frac{d}{dt})\psi(t) \cdot H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$ and $\hat{\psi}_\tau(s) = \mathcal{L}_\tau[\psi(t)]$. Then*

$$\begin{aligned} p_H(\tau - \frac{d}{ds}, s)\hat{\psi}_\tau(s) = {} & \mathcal{L}_\tau[p_H(t, \frac{d}{dt})\psi(t)] \\ & + \tau(1-\tau)\psi'(\tau) + \psi(\tau)[c - 1 - (a+b-1)\tau] + \psi(\tau)\tau(1-\tau)s. \end{aligned} \tag{95}$$

The following lemma is a special one of Lemmas 3.4 and 3.5 for the case where (47) applies.

Lemma 5.3. *Let $H_1(t)$ and $H_2(t)$ be given by* (92) *and* (93)*, and $\psi_H(t, \tau)$ for fixed $\tau > 0$ be given by*

$$\psi_H(t, \tau) := c_1(\tau) \cdot H_1(t) + c_2(\tau) \cdot H_2(t), \tag{96}$$

where $c_1(\tau)$ and $c_2(\tau)$ are constants which depend on τ and are so chosen that $\psi_H(\tau, \tau) = 0$. Then $G_H(t, 0)$ and $G_H(t, \tau)$ for $\tau > 0$, which are given by

$$G_H(t, 0) := \frac{1}{c-1} \cdot H_1(t)H(t), \quad G_H(t, \tau) := \frac{1}{\tau(1-\tau)\psi'_H(\tau, \tau)}\psi_H(t, \tau)H(t-\tau), \tag{97}$$

are the Green's functions for Equation (91)*, so that $\hat{G}_H(s, 0) = \mathcal{L}[G_H(t, 0)]$ and $\hat{G}_H(s, \tau) = \mathcal{L}_\tau[G_H(t, \tau)]$ for $\tau > 0$ satisfy*

$$p_H(\tau - \frac{d}{ds}, s)\hat{G}_H(s, \tau) = 1, \tag{98}$$

for $\tau = 0$ and $\tau > 0$, respectively.

Proof. If we put $u(t) = \frac{1}{c-1} \cdot H_1(t)$ in (94), then we have $\hat{u}(s) = \hat{G}_H(s, 0)$ on the lefthand side of (94), and the righthand side of (94) is $u(0)(c-1) = 1$, since $p_H(t, \frac{d}{dt})H_1(t) = 0$ and $H_1(0) = 1$. Thus (94) guarantees (98) for $\tau = 0$. If we put $\psi(t) = \frac{1}{\tau(1-\tau)\psi'_H(\tau,\tau)}\psi_H(t, \tau)$ and $\psi(\tau) = 0$ in (95), then we have $\hat{\psi}_\tau(s) = \hat{G}_H(s, \tau)$ on the lefthand side of (95), and the righthand side of (95) is $\tau(1-\tau)\psi'(\tau) = 1$. Thus (95) guarantees (98) for $\tau > 0$. ■

Theorem 5.1. Let $G_H(t, \tau)$ and $\psi_H(t, \tau)$ be those given in Lemma 5.3, Condition 3.1(i) be satisfied, and $u_f(t)$ be given by

$$u_f(t) = \int_0^\infty G_H(t, \tau)f(\tau)d\tau = \int_0^t \psi_H(t, \tau)\frac{f(\tau)}{\tau(1-\tau)\psi'_H(\tau, \tau)}d\tau. \tag{99}$$

Then

$$\hat{u}_f(s) := \mathcal{L}_S[u_f(t)] = \int_0^\infty \hat{G}_H(s, \tau)f(\tau)e^{-s\tau}d\tau = \int_0^\infty \hat{\psi}_H(s, \tau)\frac{f(\tau)}{\tau\psi'_H(\tau, \tau)}e^{-s\tau}d\tau. \tag{100}$$

Now $u_f(t)$ and $\hat{u}_f(s)$ are particular solutions of (91) and (94) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Proof. This is guaranteed by Corollary 3.1. ■

The condition $\psi_H(\tau,\tau) = 0$ requires that $c_1(\tau)\cdot H_1(\tau) = -c_2(\tau)\cdot H_2(\tau)$, and hence we may choose $\psi_H(t,\tau)$ as

$$\psi_H(t,\tau) = \begin{cases} H_1(t) - \frac{H_1(\tau)}{H_2(\tau)} \cdot H_2(t), & |H_1(\tau)| < |H_2(\tau)|, \\ \frac{H_2(\tau)}{H_1(\tau)} \cdot H_1(t) - H_2(t), & |H_1(\tau)| \geq |H_2(\tau)|. \end{cases} \tag{101}$$

5.2 Solution of Equation (91) in which Condition 3.1(ii) is satisfied

We give the solution of Equation (91) of which the inhomogeneous term $f(t)$ satisfies Condition 3.1(ii), so that $\hat{f}(s) = s^\beta \hat{f}_\beta(s)$, and $f_\beta(t)$ satisfies $f_\beta(t)H(t) \in \mathcal{L}^1_{loc}(\mathbb{R})$.

The following lemma is a special one of Lemma 3.9 for the case where (47) applies.

Lemma 5.4. *Let $\hat{u}(s)$ be a solution of* (94)*, and $\tilde{p}_H(-\frac{d}{ds}, s)$ be related with $p_H(-\frac{d}{ds}, s)$ given by* (90)*, by*

$$\tilde{p}_H(-\frac{d}{ds}, s) = s^{-\beta} p_H(-\frac{d}{ds}, s) s^\beta. \tag{102}$$

Then

$$\tilde{p}_H(t,s) := t(1-t)\cdot s^2 + (c - \beta - (a + b - 2\beta + 1)t)\cdot s - (a-\beta)(b-\beta), \tag{103}$$

and $\hat{w}(s) = s^{-\beta}\hat{u}(s)$ satisfies

$$\tilde{p}_H(-\frac{d}{ds}, s)\hat{w}(s) = \hat{f}_\beta(s) + u_0(c-1)s^{-\beta}. \tag{104}$$

Theorem 5.1 shows that the particular solution of (104) for the term $\hat{f}_\beta(s)$ is expressed by a particular solution of

$$\tilde{p}_H(t, \frac{d}{dt})w(t) = f_\beta(t), \quad t > 0. \tag{105}$$

Remark 5.1. We note that $\tilde{p}_H(t,s)$ given by (103) is obtained from $p_H(t,s)$ given by (90), by replacing a, b and c by $a-\beta$, $b-\beta$ and $c-\beta$, respectively, and hence the complementary solutions $H_{\beta,1}(t)$ and $H_{\beta,2}(t)$ and Green's functions $G_{\tilde{H}}(t,0)$ and $G_{\tilde{H}}(t,\tau)$ of Equation (105) are obtained from those $H_1(t)$, $H_2(t)$, $G_H(t,0)$ and $G_H(t,\tau)$, respectively, of Equation (91), by the same replacement.

Now in place of Theorem 5.1, we have the following theorem.

Theorem 5.2. Let $G_{\tilde{H}}(t,\tau)$ and $\psi_{\tilde{H}}(t,\tau)$ be obtained from $G_H(t,\tau)$ and $\psi_H(t,\tau)$ by the replacement stated in Remark 5.1, Condition 3.1(ii) be satisfied, and $w_g(t)$ be given by

$$w_g(t) = \int_0^t G_{\tilde{H}}(t,\tau) f_\beta(\tau) d\tau = \int_0^t \psi_{\tilde{H}}(t,\tau) \frac{f_\beta(\tau)}{\tau\psi'_{\tilde{H}}(\tau,\tau)} d\tau. \tag{106}$$

Then

$$\hat{w}_g(s) = \int_0^\infty \hat{G}_{\tilde{H}}(s,\tau) f_\beta(\tau) e^{-s\tau} d\tau = \int_0^\infty \hat{\psi}_{\tilde{H}}(s,\tau) \frac{f_\beta(\tau)}{\tau\psi'_{\tilde{H}}(\tau,\tau)} e^{-s\tau} d\tau. \tag{107}$$

Now $u_f(t) := {}_0D^\beta_R w_g(t)$ and $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$ are particular solutions of (91) and (94) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Proof. Theorem 5.1 states that when $w_g(t)$ is given by (106), $\hat{w}_g(s) := \mathcal{L}_S[w_g(t)]$ is the particular solution of (104) for the term $\hat{f}_\beta(s)$, and Lemma 5.4 states that $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$ is the particular solution of (94) for the term $\hat{f}(s)$. With the aid of Equation (19) in Lemma 2.3, we confirm that if $u_f(t)$ is given by $u_f(t) = {}_0D_R^\beta w_g(t)$, then $\hat{u}_f(s) = \mathcal{L}_S[u_f(t)]$ is given by $\hat{u}_f(s) = s^\beta \hat{w}_g(s)$, when $\beta \notin \mathbb{Z}_{>0}$. ■

Remark 5.2. If we put $\beta = a$ or $\beta = b$, $f_\beta(t) = 0$ and $v(t) = \frac{d}{dt}w(t)$ in (105) with (103), (105) is reduced to a homogeneous differential equation of the first order. By solving it, we obtain $v(t)$, and then $u(t) = {}_0D_R^{\beta-1}v(t)$ gives the complementary solution $H_2(t)$ of (91), see [2].

5.3 Solution of Equation (91) in which Condition 3.1(iii) is satisfied

We give the solution of Equation (91) of which the inhomogeneous term $f(t)$ satisfies Condition 3.1(iii), so that $f(t) = g_\nu(t) = \frac{1}{\Gamma(\nu)}t^{\nu-1}$ and $\hat{f}(s) = \hat{g}_\nu(s) = s^{-\nu}$. Here we use $\beta \notin \mathbb{Z}_{>-1}$ in place of $-\nu$.

Lemma 5.5. *Let $\hat{f}(s) = s^\beta$, $\tilde{p}_H(t,s)$ be given by* (103)*, and $\hat{G}_{\tilde{H}}(s,0)$ satisfy*

$$\tilde{p}_H(t,s)\hat{G}_{\tilde{H}}(s,0) = 1. \tag{108}$$

Then the particular solution of (94) *for the term $\hat{f}(s) = s^\beta$ is given by*

$$\hat{u}_f(t) = s^\beta \hat{G}_{\tilde{H}}(s,0). \tag{109}$$

Proof. $\hat{u}_f(s)$ satisfies $p_H(t,s)\hat{u}_f(t) = s^\beta$ and hence $\tilde{p}_H(t,s)s^{-\beta}\hat{u}_f(s) = 1$ by (102). Comparing this with (108), we see that (109) is satisfied. ■

By Remark 5.1, with the aid of $G_H(t,0)$ given by (97) and (92), we have

Lemma 5.6. *As the solution of* (108)*, we obtain*

$$\hat{G}_{\tilde{H}}(s,0) = \mathcal{L}_S[G_{\tilde{H}}(t,0)], \quad G_{\tilde{H}}(t,0) = \frac{1}{c-\beta-1} \cdot {}_2F_1(a-\beta, b-\beta; c-\beta; t). \tag{110}$$

Theorem 5.3. Let Condition 3.1(iii) be satisfied, and $G_{\tilde{H}}(t,0)$ be given by (110). Then the particular solution of (91) is given by

$$u_f(t) = {}_0D_R^\beta G_{\tilde{H}}(t,0) = \frac{t^{-\beta}}{(c-\beta-1)\Gamma(1-\beta)} \cdot {}_3F_2(1, a-\beta, b-\beta; 1-\beta, c-\beta; t), \tag{111}$$

where ${}_3F_2(a_1,a_2,a_3;c_1,c_2;z) = \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k(a_3)_k}{k!(c_1)_k(c_2)_k} z^k$.

Proof. We confirm that when $u_f(t)$ is given by (111), $\hat{u}_f(s) = \mathcal{L}_S[u_f(t)]$ is given by (109) and (110), with the aid of Equation (19) in Lemma 2.3. ■

6 Solution of a Fractional Equation with Coefficients of Polynomial of at Most First Degree

We now consider $p_F(t,s)$ given by

$$p_F(t,s) = ts^{3/2} + ats + bs^{1/2} + ac, \tag{112}$$

and then we give solutions of the fractional differential equation:

$$p_F(t, {}_0D_R)u(t) := t \cdot {}_0D_R^{3/2}\, u(t) + at \cdot {}_0D_R\, u(t) + b \cdot {}_0D_R^{1/2} u(t) + ac \cdot u(t) \\ = f(t), \quad t > 0. \tag{113}$$

In this section, we adopt the following condition.

Condition 6.1. Condition 1.1 with the paragraph "S is an enumerable set of $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$" replaced by "$S$ is an enumerable set of $\nu \in \{z \in \mathbb{C} | 2z \notin \mathbb{Z}_{<1}\}$".

6.1 Complementary solutions of Equation (113)

In this section, we show that the basic complementary solutions of Equation (113) are given by

$$F_1(t) = t^{-1/2} \sum_{k=0}^{\infty} \frac{(-1+2c)_k (-a)^k}{(-1+2b)_k \Gamma(\frac{k}{2} + \frac{1}{2})} t^{k/2}, \tag{114}$$

$$F_2(t) = t^{-b+1/2} \sum_{k=0}^{\infty} \frac{(-a)^k (1-2b+2c)_k}{k! \Gamma(-b + \frac{3}{2} + \frac{k}{2})} t^{k/2}. \tag{115}$$

By taking account of Lemma 2.3 and Condition 6.1, the AC-Laplace transform of Equation (113) is given by

$$p_F(-\frac{d}{ds}, s)\hat{u}(s) := -\frac{d}{ds}[(s^{3/2} + as)\hat{u}(s)] + bs^{1/2}\hat{u}(s) + ac\hat{u}(s) = \hat{f}(s) + (b-1)u_{-1/2}. \tag{116}$$

The complementary solution of this differential equation is given by

$$(s^{3/2} + as)\hat{u}(s) = Cs^b (1 + as^{-1/2})^{2b-2c}, \tag{117}$$

and hence

$$\hat{u}(s) = Cs^{b-3/2}(1 + as^{-1/2})^{2b-2c-1} = Cs^{b-3/2} \sum_{k=0}^{\infty} \frac{(-a)^k (1-2b+2c)_k}{k!} s^{-k/2}. \tag{118}$$

By the inverse AC-Laplace transform, we have $u(t) = CF_2(t)$.

The derivation of the basic complementary solutions given by (114) and (115) of Equation (113) by using the basic method is as follows.

We assume that the solution and its Laplace transform are expressed by

$$u(t) = t^{\alpha} \sum_{k=0}^{\infty} p_k \frac{t^{k/2}}{\Gamma(\alpha + \frac{k}{2} + 1)}, \tag{119}$$

$$\hat{u}(s) = s^{-\alpha-1} \sum_{k=0}^{\infty} p_k s^{-k/2}, \tag{120}$$

where $p_0 \neq 0$ and $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. Using (120) in Equation (116) with inhomogeneous term $(b-1)u_{-1/2} = 1$, we obtain

$$(\alpha - \frac{1}{2} + b)p_0 s^{-\alpha-1/2} + \sum_{k=1}^{\infty}[(\alpha + \frac{k}{2} - \frac{1}{2} + b)p_k + (\alpha + \frac{k}{2} - \frac{1}{2} + c)ap_{k-1}]s^{-\alpha-k/2-1/2} = 1. \tag{121}$$

We first give the complementary solution of this equation. Then (121) with 0 on the righthand side requires $\alpha = \frac{1}{2} - b$ and

$$\frac{k}{2}p_k + (\frac{k}{2} - b + c)ap_{k-1} = 0, \qquad k \in \mathbb{Z}_{>0}, \tag{122}$$

and hence

$$p_k = \frac{-a(k-2b-2c)}{k}p_{k-1} = \frac{(-a)^k(1-2b+2c)_k}{k!}p_0, \quad k \in \mathbb{Z}_{>0}. \tag{123}$$

Putting $\alpha = \frac{1}{2} - b$ and $p_0 = C$, and using (123) in (120) and (119), we obtain (118) and $u(t) = CF_2(t)$, respectively, when $F_2(t)$ is given by (115).

We next consider Equation (121) with the inhomogeneous term 1. Then we put $\alpha = -\frac{1}{2}$, and obtain

$$\begin{aligned} &(-1+b)p_0 = 1, \\ &(\frac{k}{2} - 1 + b)p_k + (\frac{k}{2} - 1 + c)ap_{k-1} = 0, \quad k \in \mathbb{Z}_{>0}, \end{aligned} \tag{124}$$

and hence $p_0 = \frac{1}{b-1}$ and

$$p_k = -ap_{k-1}\frac{\frac{k}{2} - 1 + c}{\frac{k}{2} - 1 + b} = (-a)^k p_0 \frac{(-1+2c)_k}{(-1+2b)_k}, \quad k \in \mathbb{Z}_{>0}. \tag{125}$$

Using these in (119), we obtain $u(t) = \frac{1}{b-1}F_1(t)$, when $F_1(t)$ is given by (114). We note that $u_{-1/2} = \frac{1}{b-1}$ for this solution $u(t)$.

6.2 Green's Function for Equation (113)

In this section, we use $u^{(\rho)}(t)$, $\hat{u}^{(\rho)}(s)$ and $\hat{u}_\tau^{(\rho)}(s)$ to represent ${}_0D_R^\rho u(t)$, $\tilde{\mathcal{L}}[{}_0D_R^\rho u(t)] = s^\rho \hat{u}(s)$ and $\mathcal{L}_\tau[{}_0D_R^\rho u(t)] = \mathcal{L}_\tau[u^{(\rho)}(t)]$, respectively.

We express Equation (113) as

$$p_F(t, {}_0D_R)u(t) := t \cdot \frac{d^2}{dt^2}u^{(-1/2)}(t) + at \cdot \frac{d}{dt}u(t) + b \cdot \frac{d}{dt}u^{(-1/2)}(t) + ac \cdot u(t) = f(t), \quad t > 0, \tag{126}$$

where $u^{(-1/2)}(t) = {}_0D_R^{-1/2}u(t)$. We then express (116) as

$$p_F(-\frac{d}{ds}, s)\hat{u}(s) := \mathcal{L}[p_F(t, {}_0D_R)u(t)] - \frac{d}{ds}\langle s^2\hat{u}^{(-1/2)}(s)\rangle_0 - a\frac{d}{ds}\langle s\hat{u}(s)\rangle_0 + b\langle s\hat{u}^{(-1/2)}(s)\rangle_0. \tag{127}$$

Following Definition 3.1, we adopt the following definition.

Definition 6.1. For Equation (126), the Green's function $G_F(t,\tau)$ for $\tau \in \mathbb{R}_{\geq 0}$ is so defined that $\hat{G}_F(s,\tau) := \mathcal{L}_\tau[G_F(t,\tau)]$ satisfies

$$p_F(\tau - \frac{d}{ds}, s)\hat{G}_F(s,\tau) = 1. \tag{128}$$

In discussing the Green's function, we use the following lemma, which corresponds to Lemma 3.2.

Lemma 6.1. *Let $\tau \in \mathbb{R}$, $\psi(t)$ be such that $\frac{d^2}{dt^2}\psi(t) \cdot H(t-\tau) \in \mathcal{L}^1_{loc}(\mathbb{R})$, $\hat{\psi}_\tau(s) := \mathcal{L}_\tau[\psi(t)]$, and $\tilde{t}$ represent $\tau - \frac{d}{ds}$. Then*

$$p_F(\tilde{t}, s)\hat{\psi}_\tau(s) = \mathcal{L}_\tau[p_F(t, {}_0D_R)\psi(t)] + \langle p_F(\tilde{t}, s)\hat{\psi}_\tau(s)\rangle_0, \tag{129}$$

where

$$\langle p_F(\tilde{t}, s)\hat{\psi}_\tau(s)\rangle_0 = \tau[\psi^{(-1/2)}(\tau)s - \psi^{(1/2)}(\tau) - a\psi(\tau)] + (b-1)\psi^{(-1/2)}(\tau). \tag{130}$$

Proof. By Lemma 3.2, from (127), we obtain (129) with

$$\langle p_F(\tilde{t}, s)\hat{\psi}_\tau(s)\rangle_0 = \tilde{t}\langle s^2\hat{\psi}^{(-1/2)}_\tau(s)\rangle_0 + a\tilde{t}\langle s\hat{\psi}_\tau(s)\rangle_0 + b\langle s\hat{\psi}^{(-1/2)}_\tau(s)\rangle_0. \tag{131}$$

where $\hat{\psi}^{(-1/2)}_\tau(s) = \mathcal{L}_\tau[{}_0D_R^{-1/2}\psi(t)]$. In obtaining (130) from (131), we use

$$\langle s^2\hat{\psi}^{(-1/2)}_\tau(s)\rangle_0 = \psi^{(-1/2)}(\tau)s - \psi^{(1/2)}(\tau), \quad \langle s\hat{\psi}_\tau(s)\rangle_0 = \psi(\tau),$$
$$\langle s\hat{\psi}^{(-1/2)}_\tau(s)\rangle_0 = \psi^{(-1/2)}(\tau), \tag{132}$$

which follows from (36). ■

Lemma 6.2. *Let $F_1(t)$ and $F_2(t)$ be given by* (114) *and* (115), *and $\psi_F(t,\tau)$ for fixed $\tau \in \mathbb{R}_{>0}$ be expressed by*

$$\psi_F(t,\tau) := c_1(\tau)F_1(t) + c_2(\tau)F_2(t), \tag{133}$$

where $c_1(\tau)$ and $c_2(\tau)$ are determined such that $\psi^{(-1/2)}_F(\tau,\tau)$, which is the value of $\psi^{(-1/2)}_F(t,\tau)$ at $t = \tau$, is equal to 0. *Then $G_F(t,0)$ and $G_F(t,\tau)$ for $\tau > 0$, which are given by*

$$G_F(t,0) = \frac{1}{b-1}F_1(t)H(t), \tag{134}$$

$$G_F(t,\tau) = -\frac{1}{\tau[\psi^{(1/2)}_F(\tau,\tau) + a\psi_F(\tau,\tau)]}\psi_F(t,\tau)H(t-\tau), \tag{135}$$

are the Green's functions for Equation (113).

Proof. If we use $\tau = 0$ and $\psi(t) = G_F(t,0)$ in (129), we see that the righthand side is equal to $(b-1)u^{(-1/2)}(0) = 1$, since $F^{(-1/2)}_1(0) = 1$ is confirmed by (114). If we use $\psi(t) = G_F(t,\tau)$ in (129), we see that the righthand side is equal to 1. ■

Remark 6.1. In the last paragraph of Section 6.1, it is shown that Equation (134) is the solution of Equation (128) for $\tau = 0$.

6.3 Particular solution of Equation (113) in terms of the Green's function

Corresponding to Corollary 3.1, we have

Theorem 6.1. Let $G_F(t,\tau)$ be defined by Definition 6.1 for Equation (113), and Condition 3.1(i) be satisfied. Then $u_f(t)$ and $\hat{u}_f(s)$ given by

$$u_f(t) := \int_0^\infty G_F(t,\tau)f(\tau)d\tau,$$
$$\hat{u}_f(s) = \int_0^\infty \hat{G}_F(s,\tau)f(\tau)e^{-s\tau}d\tau, \tag{136}$$

are particular solutions of (113) and (116) for the terms $f(t)$ and $\hat{f}(s)$, respectively.

Remark 6.2. In Section 3, we introduce $\tilde{p}_2(t,s)$ related with $p_2(t,s)$, by (63). We denote the corresponding quantity for $p_F(t,s)$, by $\tilde{p}_F(t,s)$. By using formula (66), we obtain

$$\begin{aligned}\tilde{p}_F(t,s) &:= s^{-\beta}p_F(t,s)s^{\beta} = s^{-\beta}(ts^{3/2}+ats+bs^{1/2}+ac)s^{\beta}\\ &= (ts-\beta)s^{1/2}+a(ts-\beta)+bs^{1/2}+ac\\ &= ts^{3/2}+ats+(b-\beta)s^{1/2}+a(c-\beta). \end{aligned} \tag{137}$$

We note that $\tilde{p}_F(t,s)$ is given by $p_F(t,s)$ with b and c replaced by $b-\beta$ and $c-\beta$, respectively.

Remark 6.3. When we study the problem in which Condition 3.1(ii) or 3.1(iii) applies, we use $\tilde{p}_F(t,s)$ and the equation given by

$$\tilde{p}_F(t,\frac{d}{dt})w(t) = f_\beta(t), \quad t>0. \tag{138}$$

By Lemma 3.9, we have

Lemma 6.3. *Let $\tilde{p}_F(t,s)$ be given by* (137)*, and $\hat{w}(s)$ satisfy*

$$\tilde{p}_F(-\frac{d}{ds},s)\hat{w}(s) = \hat{f}_\beta(s). \tag{139}$$

Then $\hat{u}(s)$ given by $\hat{u}(s)=s^{\beta}\hat{w}(s)$ is the particular solution of (116) *for the term $\hat{f}(s)=s^{\beta}\hat{f}_\beta(s)$.*

Definition 6.2. For Equation (138), the Green's function $G_{\tilde{F}}(t,\tau)$ for $\tau\in\mathbb{R}_{\geq 0}$ is such that $\hat{G}_{\tilde{F}}(s,\tau):=\mathcal{L}_\tau[G_{\tilde{F}}(t,\tau)]$ satisfies

$$\tilde{p}_F(\tau-\frac{d}{ds},s)\hat{G}_{\tilde{F}}(s,\tau)=1. \tag{140}$$

Remark 6.4. Taking account of Definitions 6.2 and 6.1 and Remark 6.2, we note that $G_{\tilde{F}}(t,\beta)$ and $\hat{G}_{\tilde{F}}(s,\beta)$ are $G_F(t,\beta)$ and $\hat{G}_F(s,\beta)$ with b and c replaced by $b-\beta$ and $c-\beta$, respectively.

Corresponding to Theorems 4.2 and 5.2, we have

Theorem 6.2. Let $G_{\tilde{F}}(t,\tau)$ be defined by Definition 6.2, Condition 3.1(ii) be satisfied for β satisfying $2\beta\notin\mathbb{Z}_{>0}$, and $w_g(t)$ be given by

$$w_g(t):=\int_0^t G_{\tilde{F}}(t,\tau)f_\beta(\tau)d\tau. \tag{141}$$

Then

$$\hat{w}_g(s):=\mathcal{L}[w_g(t)]=\int_0^\infty \hat{G}_{\tilde{F}}(s,\tau)f_\beta(\tau)e^{-\tau s}d\tau. \tag{142}$$

Now $\hat{u}_f(s):=s^{\beta}\hat{w}_g(s)$ is the particular solution of (116) for the term $\hat{f}(s)=s^{\beta}\hat{f}_\beta(s)$, and $u_f(t)={}_0D_R^{\beta}w_g(t)$ is a particular solution of (113) for the term $f(t)={}_0D_R^{\beta}f_\beta(t)$.

Proof. By Theorem 6.1, $\hat{w}_g(s)$ given above is the solution of (139), and hence we complete the proof with the aid of Lemmas 6.3 and 3.9. ■

Theorem 6.3. Let $\beta\notin\mathbb{Z}_{>-1}$, Condition 3.1(iii) be satisfied for β satisfying $2\beta\notin\mathbb{Z}_{>0}$, $\tilde{p}_F(t,s)$ be given by (137), and $G_{\tilde{F}}(t,0)$ be defined by Definition 6.2. Then the particular solution of (116) for the term $\hat{f}(s)=s^{\beta}$ is given by

$$\hat{u}_f(s)=s^{\beta}\hat{G}_{\tilde{F}}(s,0), \tag{143}$$

and $u_f(t)={}_0D_R^{\beta}G_{\tilde{F}}(t,0)$ is a particular solution of (113) for the term $f(t)=g_{-\beta}(t)=\frac{1}{\Gamma(-\beta)}t^{-\beta-1}$.

Proof. When $\hat{f}_\beta(s)=1$, the solution $\hat{w}(s)$ of (139) is given by $\hat{w}(s)=\hat{G}_{\tilde{F}}(s,0)$ by Definition 6.2, and hence we complete the proof with the aid of Lemmas 6.3 and 3.9. ■

7 Conclusion

In the present paper, we are conserned with differential equations which are satisfied by a function $u(t)$ for $t \in \mathbb{R}_{>0}$.

We first consider a differential equation with constant coefficients. If the solution $u(t)$ has the Laplace transform $\hat{u}(s)$, the solution is obtained by taking the Laplace transform of the equation, solving it for $\hat{u}(s)$, and then taking the inverse Laplace transform of it. Even when this is not applicable, we can solve it by the methods of operational calculus or distribution theory. When the equation is inhomogeneous, the Green's function is used in distribution theory [15, 16, 11, 17]. In [3], the solutions obtained by distribution theory were shown to be also obtained by using the AC-Laplace transform.

We next consider a homogeneous differential equation with polynomial coefficients. If the solution $u(t)$ has the Laplace transform $\hat{u}(s)$, the solution is obtained by taking the Laplace transform of the equation. Even when this is not applicable, we can solve it by the methods of operational calculus [8, 9, 7] or distribution theory [6, 18]. In [1, 2], the solutions obtained by distribution theory were shown to be also obtained by using the AC-Laplace transform.

We finally consider an inhomogeneous differential equation with polynomial coefficients. In [3], the solution was shown to be obtained by the methods of distribution theory with the aid of the Green's function; see also [16, 19]. In the present paper, the solutions obtained by distribution theory are also obtained by using the AC-Laplace transform.

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Competing Interests

Authors have declared that no competing interests exist.

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