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Common Fixed Points of Kannan and Chatterjea Types of Mappings in a Complete Metric Space

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

This paper contains several generalizations of the theorems for common fixed point of R. Kannan, S. K. Chatterjea and P. V. Koparde & B. B. Waghmode types of mappings. These generalizations are done by using a sequentially convergent mappings. Trough several examples, we have shown that the generalized claims are inapplicable, and that the obtained generalized claims prove the existence of a unique common fixed point of considered mappings.

Keywords: Common fixed point; complete metric space.

1 Introduction

The development of the theory of fixed point started with the S. Banach [1] theorem, which actually consider the principle of the contractive mapping. The above theorem was presented by Banach, as a part of his doctor dissertation and is a very important researching instrument in many fields of mathematics.

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R. Kannan [2], 1968 and S. K. Chatterjea [3], 1972, proved that if (X,d) is a complete metric space and $S:X\to X$ is mapping so that it exists $\alpha\in(0,\frac{1}{2})$ and for all $x,y\in X$ respectively, one of the following inequalities is satisfied:

$$d(Sx,Sy)\le\alpha(d(x,Sx)+d(y,Sy)),$$

$$d(Sx,Sy)\le\alpha(d(x,Sy)+d(y,Sx)),$$

then, S has a unique fixed point. In [4] S. Moradi and D. Alimohammadi generalized R. Kannan result, by using the sequentially convergent mappings, in [5] are proven several generalizations of R. Kannan, S. K. Chatterjea, P. V. Koparde and B. B. Waghmode theorems [6] based on sequentially convergent, defined as the following:

Definition 1 [7]. Let (X,d) be a metric space. A mapping $T:X\to X$ is said sequentially convergent if, for each sequence $\{y_n\}$ the following holds true:

if $\{Ty_n\}$ convergences, then $\{y_n\}$ also convergences.

2 Common Fixed Points for Kannan Type of Mappings

Theorem 1. Let (X,d) be a complete metric space, $T:X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2:X\to X$. If there exist $\alpha>0,\beta\ge0$ such that $2\alpha+\beta<1$ and

$$d(TS_1x,TS_2y)\le\alpha(d(Tx,TS_1x)+d(Ty,TS_2y))+\beta d(Tx,Ty), \tag{1}$$

for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Let x_0 be any point in X and let the sequence $\{x_n\}$ be defined as the following

$$x_{2n+1}=S_1x_{2n},\ \ x_{2n+2}=S_2x_{2n+1},\ \text{ for } n=0,1,2,\dots.$$

If it exists $n\ge0$, so that $x_n=x_{n+1}=x_{n+2}$, then it is easy to be proven that $u=x_n$ is a common fixed point for S_1 and S_2 . Therefore, let's suppose that there does not exist three consecutive identical terms of the sequence $\{x_n\}$. Then, by using the inequality (1), it is easy to be proven that for each $n\ge1$, the following holds true

$$d(Tx_{2n+1},Tx_{2n})\le\alpha[d(Tx_{2n+1},Tx_{2n})+d(Tx_{2n},Tx_{2n-1})]+\beta d(Tx_{2n},Tx_{2n-1}) \text{ and}$$

$$d(Tx_{2n-1},Tx_{2n})\le\alpha d(Tx_{2n-2},Tx_{2n-1})+\alpha d(Tx_{2n-1},Tx_{2n})+\beta d(Tx_{2n-2},Tx_{2n-1}).$$

The latter implies that for each $n=0,1,2,\dots$ and $\lambda=\frac{\alpha+\beta}{1-\alpha}<1$

$$d(Tx_{n+1},Tx_n)\le\lambda d(Tx_n,Tx_{n-1}), \tag{2}$$

holds true. Thus, the inequality (2) implies that

$$d(Tx_{n+1},Tx_n) \le \lambda^n d(Tx_1,Tx_0)\,, \tag{3}$$

for each $n=0,1,2,...$. Further, (3) implies that for all $m,n\in \mathbf{N}$, $n>m$:

$$d(Tx_n,Tx_m) \le \tfrac{\lambda^m}{1-\lambda} d(Tx_1,Tx_0)$$

holds true. That is the sequence $\{Tx_n\}$ is Caushy. But, X is a complete metric space, therefore the sequence $\{Tx_n\}$ is convergent. Further, the mapping $T:X\to X$ is sequentially convergent and since $\{Tx_n\}$ is convergent, it is true that the sequence $\{x_n\}$ is convergent, i.e. it exists $u\in X$ so that $\lim\limits_{n\to\infty} x_n = u$. Thus, the continuous of T implies that $\lim\limits_{n\to\infty} Tx_n = Tu$. So,

$$\begin{aligned} d(Tu,TS_1u) &\le d(Tu,Tx_{2n+2}) + d(Tx_{2n+2},TS_1u) = d(Tu,Tx_{2n+2}) + d(TS_2x_{2n+1},TS_1u) \\ &\le d(Tu,Tx_{2n+2}) + \alpha(d(Tu,TS_1u) + d(Tx_{2n+1},TS_2x_{2n+1})) + \beta d(Tu,Tx_{2n+1}) \\ &\le d(Tu,Tx_{2n+2}) + \alpha(d(Tu,TS_1u) + d(Tx_{2n+1},Tx_{2n+2})) + \beta d(Tu,Tx_{2n+1}). \end{aligned}$$

For $n\to\infty$, the latter implies that $d(Tu,TS_1u) \le \alpha d(Tu,TS_1u)$, holds true. But, $\alpha<1$, so, $d(TS_1u,Tu)=0$. That is, $TS_1u=Tu$ and since T is injection, we get that $S_1u=u$, i.e. u is a fixed point for S_1. Analogously, u is a fixed point for S_2. We will prove that S_1 and S_2 have a unique common fixed point. Let $v\in X$ be one other fixed point for S_2, i.e. $S_2v=v$. So,

$$d(Tu,Tv) = d(TS_1u,TS_2v) \le \alpha(d(Tu,TS_2v) + d(Tv,TS_1u)) + \beta d(Tu,Tv) = (2\alpha+\beta)d(Tu,Tv),,$$

Since, $2\alpha+\beta<1$ we get that $d(Tu,Tv)\le 0$, therefore $Tu=Tv$. But, T is injection, therefore $u=v$. ■

Consequence 1. Let (X,d) be a complete metric space, $T:X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2:X\to X$. If it exists $\lambda\in(0,1)$ so that

$$d(TS_1x,TS_2y) \le \lambda\sqrt[3]{d(Tx,TS_1x)\cdot d(Ty,TS_2y)\cdot d(Tx,Ty)}$$

for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The arithmetic-geometric mean inequality implies that

$$d(TS_1x,TS_2y) \le \tfrac{\lambda}{3}(d(Tx,TS_1x) + d(Ty,TS_2y) + d(Tx,Ty)).$$

holds true. Thus, for $\alpha=\beta=\frac{\lambda}{3}$ in Theorem 1, we get the required claim.■

Consequence 2. Let (X,d) be a complete metric space, $T:X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2:X\to X$. If there exist $\alpha>0, \beta\ge 0$ so that $2\alpha+\beta<1$ and

$$d(TS_1x,TS_2y) \le \alpha\frac{d^2(Tx,TS_1x)+d^2(Ty,TS_2y)}{d(Tx,TS_1x)+d(Ty,TS_2y)} + \beta d(Tx,Ty)\,,$$

holds true for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The inequality given in the condition implies (1). Thus, the claim is directly implied by Theorem 1. ■

Consequence 3. Let (X,d) be a complete metric space, $T: X \to X$ is continuous, injection and sequentially convergent mapping and $S_1, S_2 : X \to X$. If it exists $\alpha \in (0, \frac{1}{2})$ so that

$$d(TS_1x, TS_2y) \le \alpha(d(Tx, TS_1x) + d(Ty, TS_2y))$$

holds true for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $\beta = 0$ in the Theorem 1, we get the required claim. ■

Consequence 4. Let (X,d) be a complete metric space and $S_1, S_2 : X \to X$. If there exist $\alpha > 0, \beta \ge 0$ so that $2\alpha + \beta < 1$ and

$$d(TS_1x, TS_2y) \le \alpha(d(Tx, TS_1x) + d(Ty, TS_2y))$$

holds true for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $Tx = x$ in Theorem 1 we get the required claim t. ■

Consequence 5. Let (X,d) be a complete metric space and $S_1, S_2 : X \to X$. If there exists $\alpha \in (0, \frac{1}{2})$ so that

$$d(S_1x, S_2y) \le \alpha(d(x, S_1x) + d(y, S_2y))$$

holds true for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $Tx = x$ in Consequence 3 or $\beta = 0$ in Consequence 4 we obtain the required claim. ■

Consequence 6. Let (X,d) be a complete metric space, $T: X \to X$ be continuous, injection and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $p, q \in \mathbf{N}$, $\alpha > 0, \beta \ge 0$ so that $2\alpha + \beta < 1$ and

$$d(TS_1^p x, TS_2^q y) \le \alpha(d(Tx, TS_1^p x) + d(Ty, TS_2^q y)) + \beta d(Tx, Ty)$$

holds true for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Since the Theorem 1, the mappings S_1^p and S_2^q have a unique common fixed point $u \in X$. That is, $S_1^p u = u$. Therefore, $S_1 u = S_1(S_1^p u) = S_1^p(S_1 u)$ that is, $S_1 u$ is a fixed point for S_1^p. Analogously, thereby $S_2^q u = u$ it is true that $S_2 u = S_2(S_2^q u) = S_2^q(S_2 u)$ that is, $S_2 u$ is a fixed point for S_2^q. But, the proof of Theorem 1 implies that the both S_2^q and S_1^p have a unique fixed point. Therefore, $u = S_2 u$ and $u = S_1 u$. So, $u \in X$ is a common fixed point for S_1 and S_2. If $v \in X$ is one other fixed point for S_1 and S_2, then it is a common fixed point for S_1^p and S_2^q. But, S_1^p and S_2^q have a unique fixed point. Therefore, $v = u$. ■

Example 1. Let $X=\{0\}\cup\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$ and d be an Euclidian metric in X. Then, (X,d) is a complete metric space. Let the mappings $S_1,S_2:X\to X$ be determined by $S_1(0)=S_2(0)=0$ and $S_1(\frac{1}{n})=\frac{1}{n+1}$, $S_2(\frac{1}{n})=\frac{1}{n+2}$, for $n\geq 1$. If there exists $\alpha\in(0,\frac{1}{2})$, such that, for all $x,y\in X$ the condition given in Consequence 5 is satisfied, then for $x=\frac{1}{n-1}$, $y=\frac{1}{2n-2}$ we get that for each $n>1$, $\frac{n-1}{3}\leq\alpha<\frac{1}{2}$ has to be satisfied, which is contradictory. Thus, it is impossible to apply the Consequence 5. The mapping $T:X\to X$ determined by $T(0)=0$ and $T(\frac{1}{n})=\frac{1}{[e^{2n}]}$, for $n\geq 1$ is continuous, injection and sequentially convergent mapping. Further, since $[x]\cdot[y]\leq[xy]$, for all $x,y\geq 0$ we get that for each $n\geq 1$ the following holds true

$$7[e^{2n}]\leq[e^2]\cdot[e^{2n}]=[e^2e^{2n}]=[e^{2(n+1)}],$$

i.e.

$$\frac{1}{[e^{2(n+1)}]}\leq\frac{1}{6}(\frac{1}{[e^{2n}]}-\frac{1}{[e^{2(n+1)}]}).$$

Therefore, for all $m,n\in\mathbf{N}$, $m>n$ the following holds true

$$\begin{aligned}|TS_1(\tfrac{1}{n})-TS_2(\tfrac{1}{m})|&=|\tfrac{1}{[e^{2(n+1)}]}-\tfrac{1}{[e^{2(m+2)}]}|<\tfrac{1}{[e^{2(n+1)}]}+\tfrac{1}{[e^{2(m+2)}]}\leq\tfrac{1}{[e^{2(n+1)}]}+\tfrac{1}{[e^{2(m+1)}]}\\&\leq\tfrac{1}{6}|\tfrac{1}{[e^{2n}]}-\tfrac{1}{[e^{2(n+1)}]}|+\tfrac{1}{6}|\tfrac{1}{[e^{2m}]}-\tfrac{1}{[e^{2(m+1)}]}|\leq\tfrac{1}{6}|\tfrac{1}{[e^{2n}]}-\tfrac{1}{[e^{2(n+1)}]}|+\tfrac{1}{6}|\tfrac{1}{[e^{2m}]}-\tfrac{1}{[e^{2(m+2)}]}|\\&=\tfrac{1}{6}[|T(\tfrac{1}{n})-TS_1(\tfrac{1}{n})|+|T(\tfrac{1}{m})-TS_2(\tfrac{1}{m})|].\end{aligned}$$

For each $n\in\mathbf{N}$, the following also holds true

$$\begin{aligned}|TS_1(0)-TS_2(\tfrac{1}{n})|&=\tfrac{1}{[e^{2(n+2)}]}\leq\tfrac{1}{[e^{2(n+1)}]}\leq\tfrac{1}{6}|\tfrac{1}{[e^{2n}]}-\tfrac{1}{[e^{2(n+1)}]}|\leq\tfrac{1}{6}|\tfrac{1}{[e^{2n}]}-\tfrac{1}{[e^{2(n+2)}]}|=\\&\tfrac{1}{6}[|T(0)-TS_1(0)|+|T(\tfrac{1}{n})-TS_2(\tfrac{1}{n})|],\end{aligned}$$

$$|TS_2(0)-TS_1(\tfrac{1}{n})|=\tfrac{1}{[e^{2(n+1)}]}\leq\tfrac{1}{6}|\tfrac{1}{[e^{2n}]}-\tfrac{1}{[e^{2(n+1)}]}|=\tfrac{1}{6}[|T(0)-TS_2(0)|+|T(\tfrac{1}{n})-TS_1(\tfrac{1}{n})|].$$

So, the condition given in Consequence 3 is satisfied for $\alpha=\frac{1}{6}$. Therefore, the mappings S_1 and S_2 have a unique common fixed point. ■

3 Common Fixed Points for Chatterjea Type of Mappings

Theorem 2. Let (X,d) be a complete metric space, $T:X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2:X\to X$. If there exist $p,q\in\mathbf{N}$, $\alpha>0,\beta\geq 0$ so that $2\alpha+\beta<1$ and

$$d(TS_1x,TS_2y)\leq\alpha(d(Tx,TS_2y)+d(Ty,TS_1x))+\beta d(Tx,Ty), \tag{4}$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Let x_0 be any point in X and let the sequence $\{x_n\}$ be defined as the following $x_{2n+1} = S_1 x_{2n}$, $x_{2n+2} = S_2 x_{2n+1}$, for $n = 0,1,2,...$. If there exists $n \geq 0$, so that $x_n = x_{n+1} = x_{n+2}$, then it is easy to be proven that $u = x_n$ is a common fixed point for S_1 and S_2. Therefore, let's suppose that there does not exist three consecutive identical terms of the sequence $\{x_n\}$. Then, by using (4), it is easy to be proven that for each $n \geq 1$ the following holds true

$$d(Tx_{2n+1}, Tx_{2n}) \leq \alpha d(Tx_{2n-1}, Tx_{2n}) + \alpha d(Tx_{2n}, Tx_{2n+1}) + \beta d(Tx_{2n}, Tx_{2n-1}),$$

$$d(Tx_{2n-1}, Tx_{2n}) \leq \alpha d(Tx_{2n-2}, Tx_{2n-1}) + \alpha d(Tx_{2n-1}, Tx_{2n}) + \beta d(Tx_{2n-2}, Tx_{2n-1}).$$

The latter implies that

$$d(Tx_{n+1}, Tx_n) \leq \lambda d(Tx_n, Tx_{n-1}), \tag{5}$$

holds true, for each $n = 0,1,2,...$, and $\lambda = \frac{\alpha+\beta}{1-\alpha} < 1$. The inequality (5) implies

$$d(Tx_{n+1}, Tx_n) \leq \lambda^n d(Tx_1, Tx_0), \tag{6}$$

for each $n = 0,1,2,...$. Further, by using (6), analogously as the proof of the Theorem 1, we get that the sequence $\{Tx_n\}$ is convergent, therefore the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in X$ so that $\lim\limits_{n\to\infty} x_n = u$ and $\lim\limits_{n\to\infty} Tx_n = Tu$. We will prove that u is a fixed point for S_1. We have

$$\begin{aligned} d(Tu, TS_1u) &\leq d(Tu, Tx_{2n+2}) + d(Tx_{2n+2}, TS_1u) = d(Tu, Tx_{2n+2}) + d(TS_2x_{2n+1}, TS_1u) \\ &\leq d(Tu, Tx_{2n+2}) + \alpha(d(Tx_{2n+1}, TS_1u) + d(Tu, TS_2x_{2n+1})) + \beta d(Tu, Tx_{2n+1}) \\ &\leq d(Tu, Tx_{2n+2}) + \alpha(d(Tx_{2n+1}, TS_1u) + d(Tu, Tx_{2n+2})) + \beta d(Tu, Tx_{2n+1}). \end{aligned}$$

For $n \to \infty$, we get that $d(Tu, TS_1u) \leq \alpha d(Tu, TS_1u)$ But, $\alpha < 1$, and the latter implies that $d(TS_1u, Tu) = 0$. Then, analogously as the proof of Theorem 1,we get that u is a fixed point for S_1. Analogously, u is a fixed point for S_2. We will prove that S_1 and S_2 have a unique common fixed point. Let $v \in X$ be one other fixed point for S_2, i.e. $S_2 v = v$. We have

$$d(Tu, Tv) = d(TS_1u, TS_2v) \leq \alpha(d(Tu, TS_2v) + d(Tv, TS_1u)) + \beta d(Tu, Tv) = (2\alpha + \beta) d(Tu, Tv),$$

and since $2\alpha + \beta < 1$ we get that $d(Tu, Tv) = 0$, therefore $Tu = Tv$. But, T is injection. Therefore, $u = v$. ■

Consequence 7. Let (X, d) be a complete metric space, $T : X \to X$ be continuous, injection and sequentially convergent mapping and $S_1, S_2 : X \to X$. If $\lambda \in (0,1)$, and

$$d(TS_1x, TS_2y) \leq \lambda \sqrt[3]{d(Tx, TS_2y) \cdot d(Ty, TS_1x) \cdot d(Tx, Ty)}$$

holds true, for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The arithmetic-geometric mean inequality implies

$$d(TS_1x,TS_2y) \le \frac{\lambda}{3}(d(Tx,TS_2y)+d(Ty,TS_1x)+d(Tx,Ty)).$$

For $\alpha=\beta=\frac{\lambda}{3}$, the Theorem 2 implies the required claim. ■

Consequence 8. Let (X,d) be a complete metric space, $T: X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2 : X\to X$. If there exist $\alpha>0,\beta\ge 0$ so that $2\alpha+\beta<1$ and

$$d(TS_1x,TS_2y) \le \alpha\frac{d^2(Tx,TS_2y)+d^2(Ty,TS_1x)}{d(Tx,TS_2y)+d(Ty,TS_1x)}+\beta d(Tx,Ty)$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The condition inequality implies (4). Further, the required claim is implied by Theorem 2. ■

Consequence 9. Let (X,d) be a complete metric space, $T: X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2 : X\to X$. If there exists $\alpha\in(0,\frac{1}{2})$ and

$$d(TS_1x,TS_2y) \le \alpha(d(Tx,TS_2y)+d(Ty,TS_1x))$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $\beta=0$ in Theorem 2, we get the required claim. ■

Consequence 10. Let (X,d) be a complete metric space and $S_1,S_2 : X\to X$. If there exist $\alpha>0,\beta\ge 0$, so that $2\alpha+\beta<1$ and

$$d(S_1x,S_2y) \le \alpha(d(x,S_2y)+d(y,S_1x))+\beta d(x,y)$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $Tx=x$ in Theorem 2, we get the required claim. ■

Consequence 11. Let (X,d) be a complete metric space and $S_1,S_2 : X\to X$. If there exists $\alpha\in(0,\frac{1}{2})$ so that

$$d(S_1x,S_2y) \le \alpha(d(x,S_2y)+d(y,S_1x)),$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Either for $Tx=x$ in Consequence 9 or $\beta=0$ in Consequence 10, we get the required claim. ■

Consequence 12. Let (X,d) be a complete metric space, $T: X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2 : X\to X$. If there exist $p,q\in \mathbf{N}$, $\alpha>0,\beta\ge 0$ so that $2\alpha+\beta<1$ and

$$d(TS_1^p x,TS_2^q y) \le \alpha(d(Tx,TS_2^q y)+d(Ty,TS_1^p x))+\beta d(Tx,Ty),$$

holds true, for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The proof is identical to the proof of Consequence 6. ■

Example 2. Let (X,d) and $T, S_1, S_2 : X \to X$ be the metric space and the mappings defined as in the Example 1, respectively. If there exists $\alpha \in (0, \frac{1}{2})$ so that for all $x, y \in X$ the condition given in the Consequence 11 is satisfied, then for $x = \frac{1}{n-1}$, $y = \frac{1}{2n-2}$ we get that for each $n > 1$ the following has to be satisfied $\frac{n-1}{2n-1} \le \alpha < \frac{1}{2}$. The latter is contradictory, thereby the sequence $\{\frac{n-1}{2n-1}\}$ convergences to $\frac{1}{2}$. Therefore, Consequence 11 is not applicable for the example above. On the other hand,

$$\begin{aligned}|TS_1(\tfrac{1}{n}) - TS_2(\tfrac{1}{m})| &= |\tfrac{1}{[e^{2(n+1)}]} - \tfrac{1}{[e^{2(m+2)}]}| < \tfrac{1}{[e^{2(n+1)}]} \le \tfrac{1}{6}(\tfrac{1}{[e^{2n}]} - \tfrac{1}{[e^{2(n+1)}]}) \\ &\le \tfrac{1}{6}[\tfrac{1}{[e^{2n}]} - \tfrac{1}{[e^{2(n+1)}]} + \tfrac{1}{[e^{2m}]} - \tfrac{1}{[e^{2(m+2)}]}] \le \tfrac{1}{6}[|\tfrac{1}{[e^{2n}]} - \tfrac{1}{[e^{2(m+2)}]}| + |\tfrac{1}{[e^{2m}]} - \tfrac{1}{[e^{2(n+1)}]}|] \\ &= \tfrac{1}{6}[|T(\tfrac{1}{n}) - TS_2(\tfrac{1}{m})| + |T(\tfrac{1}{m}) - TS_1(\tfrac{1}{n})|],\end{aligned}$$

holds true for all $m, n \in \mathbf{N}$, and also

$$|TS_1(0) - TS_2(\tfrac{1}{n})| = \tfrac{1}{[e^{2(n+2)}]} \le \tfrac{1}{6}(\tfrac{1}{[e^{2n}]} + \tfrac{1}{[e^{2(n+2)}]}) = \tfrac{1}{6}[|T(0) - TS_2(\tfrac{1}{n})| + |T(\tfrac{1}{n}) - TS_1(0)|],$$

$$|TS_2(0) - TS_1(\tfrac{1}{n})| = \tfrac{1}{[e^{2(n+1)}]} \le \tfrac{1}{6}(\tfrac{1}{[e^{2n}]} + \tfrac{1}{[e^{2(n+1)}]}) = \tfrac{1}{6}[|T(0) - TS_1(\tfrac{1}{n})| + |T(\tfrac{1}{n}) - TS_2(0)|],$$

holds true for each $n \in \mathbf{N}$. So, for $\alpha = \frac{1}{6}$ the condition given in Consequence 9 is satisfied, i.e. S_1 and S_2 have a unique common fixed point. ■

4 Common Fixed Points for Koparde-Waghmode Type of Mappings

Theorem 3. Let (X,d) be a complete metric space, $T : X \to X$ be continuous, injection and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $\alpha > 0, \beta \ge 0$ so that $2\alpha + \beta < 1$ and

$$d^2(TS_1x, TS_2y) \le \alpha(d^2(Tx, TS_1x) + d^2(Ty, TS_2y)) + \beta d^2(Tx, Ty), \tag{7}$$

holds true, for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Let x_0 be any point in X and the sequence $\{x_n\}$ be defined as the following $x_{2n+1} = S_1 x_{2n}$, $x_{2n+2} = S_2 x_{2n+1}$, for $n = 0, 1, 2, \ldots$. If there exists $n \ge 0$, so that $x_n = x_{n+1} = x_{n+2}$ holds true, then it is easy to be proven that $u = x_n$ is a common fixed point of S_1 and S_2. Therefore, let's suppose that there no exists three consecutive identical terms of the sequence $\{x_n\}$. Then, by using (7), we prove that for each $n \ge 1$ the following holds true

$$d^2(Tx_{2n+1}, Tx_{2n}) \le \alpha(d^2(Tx_{2n}, Tx_{2n+1}) + d^2(Tx_{2n-1}, Tx_{2n})) + \beta d^2(Tx_{2n}, Tx_{2n-1}),$$

$$d^2(Tx_{2n-1}, Tx_{2n}) \le \alpha(d^2(Tx_{2n-2}, Tx_{2n-1}) + d^2(Tx_{2n-1}, Tx_{2n})) + \beta d^2(Tx_{2n-2}, Tx_{2n-1}).$$

The latter implies the following

$$d(Tx_{n+1},Tx_n)\le \lambda d(Tx_n,Tx_{n-1}),\tag{8}$$

for each $n=0,1,2,...$, and $\lambda=\sqrt{\frac{\alpha+\beta}{1-\alpha}}<1$. Further, (8) implies the following

$$d(Tx_{n+1},Tx_n)\le \lambda^n d(Tx_1,Tx_0),\tag{9}$$

for each $n=0,1,2,...$. By applying (8), analogously to Theorem 1,we get that the sequence $\{Tx_n\}$ is convergent. Therefore, $\{x_n\}$ is also convergent, i.e. it exists $u\in X$ so that $\lim\limits_{n\to\infty} x_n=u$ and $\lim\limits_{n\to\infty} Tx_n=Tu$. We will prove that u is a fixed point for S_1. We have

$$\begin{aligned} d(Tu,TS_1u) &\le d(Tu,Tx_{2n+2})+d(Tx_{2n+2},TS_1u)=d(Tu,Tx_{2n+2})+d(TS_1u,TS_2x_{2n+1})\\ &\le d(Tu,Tx_{2n+2})+\sqrt{\alpha(d^2(Tu,TS_1u)+d^2(Tx_{2n+1},TS_2x_{2n+1}))+\beta d^2(Tu,Tx_{2n+1})}\\ &= d(Tu,Tx_{2n+2})+\sqrt{\alpha(d^2(Tu,TS_1u)+d^2(Tx_{2n+1},Tx_{2n+2}))+\beta d^2(Tu,Tx_{2n+1})} \end{aligned}$$

holds true for each $n\in\mathbf{N}$. For $n\to\infty$, we get that $d(Tu,TS_1u)\le\sqrt{\alpha}d(Tu,TS_1u)$. But, $\sqrt{\alpha}<1$, and therefore $d(Tu,TS_1u)=0$. Analogously to Theorem 1, u is a fixed point for S_1, and also u is a fixed point for S_2. We will prove that S_1 and S_2 have a unique common fixed point. Let $v\in X$ be one other fixed point for S_2, i.e. $S_2v=v$. We have

$$d^2(Tu,Tv)=d^2(TS_1u,TS_2v)\le\alpha(d^2(Tu,TS_1u)+d^2(Tv,TS_2v))+\beta d^2(Tu,Tv)=\beta d^2(Tu,Tv).$$

Since $0\le\beta<1$, we get that $d(Tu,Tv)=0$. Therefore, $Tu=Tv$. But, T is injection, and therefore, $u=v$. ■

Consequence 13. Let (X,d) be a complete metric space, $T:X\to X$ be continuous, injection and sequentially convergent mapping and $S_1,S_2:X\to X$. If there exists $\alpha\in(0,\frac{1}{2})$ and

$$d^2(TS_1x,TS_2y)\le\alpha(d^2(Tx,TS_1x)+d^2(Ty,TS_2y)),$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $\beta=0$ in Theorem 3, we obtain the required claim. ■

Consequence 14. Let (X,d) be a complete metric space and $S_1,S_2:X\to X$. If there exist $\alpha>0,\beta\ge0$ so that $2\alpha+\beta<1$ and

$$d^2(S_1x,S_2y)\le\alpha(d^2(x,S_1x)+d^2(y,S_2y))+\beta d^2(x,y),$$

holds true, for all $x,y\in X$, then S_1 and S_2 have a unique common fixed point.

Proof. For $Tx=x$ in Theorem 3, we obtain the required claim. ■

Consequence 15. Let (X,d) be a complete metric space and $S_1, S_2 : X \to X$. If there exists $\alpha \in (0, \frac{1}{2})$ so that

$$d^2(S_1 x, S_2 y) \le \alpha(d^2(x, S_1 x) + d^2(y, S_2 y))$$

holds true, for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. Either for $Tx = x$ in Consequence 13 or for $\beta = 0$ in Consequence 14, we get the required claim. ■

Consequence 16. Let (X,d) be a complete metric space, $T : X \to X$ be continuous, injection and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $p, q \in \mathbf{N}$, $\alpha > 0, \beta \ge 0$ so that $2\alpha + \beta < 1$ and

$$d^2(TS_1^p x, TS_2^q y) \le \alpha(d^2(Tx, TS_1^p x) + d^2(Ty, TS_2^q y)) + \beta^2 d(Tx, Ty)$$

holds true, for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Proof. The proof is identical to the proof of Consequence 6. ■

Example 3. Let (X,d) and $T, S_1, S_2 : X \to X$ be the metric space and the mappings defined as in the Example 1, respectively. If there exists $\alpha \in (0, \frac{1}{2})$ so that, for all $x, y \in X$ the condition given in Consequence 16 is satisfied, then for $x = \frac{1}{n-1}$, $y = \frac{1}{2n-2}$ we get that for each $n > 1$ the following has to be satisfied $\frac{(n-1)^2}{5} \le \alpha < \frac{1}{2}$, which is not possible. So, for this case it is impossible to use Consequence 15. Further, T is continuous, injection and sequentially convergent mapping and

$$\begin{aligned} |TS_1(\tfrac{1}{n}) - TS_2(\tfrac{1}{m})|^2 &\le |TS_1(\tfrac{1}{n}) + TS_2(\tfrac{1}{m})|^2 \le 2(|TS_1(\tfrac{1}{n})|^2 + |TS_2(\tfrac{1}{m})|^2) \\ &\le 2 \cdot (\tfrac{1}{36}|T(\tfrac{1}{n}) - TS_1(\tfrac{1}{n})|^2 + \tfrac{1}{36}|T(\tfrac{1}{m}) - TS_2(\tfrac{1}{m})|^2), \\ &= \tfrac{1}{18}(|T(\tfrac{1}{n}) - TS_1(\tfrac{1}{n})|^2 + |T(\tfrac{1}{m}) - TS_2(\tfrac{1}{m})|^2) \end{aligned}$$

holds true for all $m, n \in \mathbf{N}$, and also

$$|TS_1(0) - TS_2(\tfrac{1}{n})|^2 = |TS_2(\tfrac{1}{n})|^2 \le \tfrac{1}{36}|T(\tfrac{1}{n}) - TS_2(\tfrac{1}{n})|^2 < \tfrac{1}{18}(|T(0) - TS_1(0)|^2 + |T(\tfrac{1}{n}) - TS_2(\tfrac{1}{n})|^2),$$

$$|TS_2(0) - TS_1(\tfrac{1}{n})|^2 = |TS_1(\tfrac{1}{n})|^2 \le \tfrac{1}{36}|T(\tfrac{1}{n}) - TS_1(\tfrac{1}{n})|^2 < \tfrac{1}{18}(|T(0) - TS_2(0)|^2 + |T(\tfrac{1}{n}) - TS_1(\tfrac{1}{n})|^2),$$

holds true for each $n \in \mathbf{N}$. That is, for $\alpha = \frac{1}{18}$ the condition given in Consequence 13 is satisfied, i.e. S_1 and S_2 have a unique common fixed point. ■

5 Conclusion

In this paper we have proven several claims about common fixed points for two mappings. The proven claims are generalizations of already known theorems, which are not applicable for certain cases. Logically, the question whether analogous generalizations can be used in other cases is asked. For example, whether analogously the claims about the fixed points for the 2-Banach spaces proven in [8], can be generalized?

Competing Interests

Authors have declared that no competing interests exist.

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