British Journal of Mathematics & Computer Science
11(4): 1-7, 2015, Article no.BJMCS.19677
ISSN: 2231-0851

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Evolution Operator of the Bipartite Orbit of Graphs

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Article Information

DOI: 10.9734/BJMCS/2015/19677
Editor(s):
(1) Sheng Zhang, Department of Mathematics, Bohai University, Jinzhou, China.
Reviewers:
(1) Anonymous, Vidya Academy of Science and Technology, India.
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(4) Krasimir Yordzhev, South-West University, Bulgaria.
Complete Peer review History: http://sciencedomain.org/review-history/11299

Original Research Article

Received: 23 June 2015
Accepted: 09 August 2015
Published: 06 September 2015

Abstract

The operation of substitution consists of replacing a vertex of a graph by another graph. This new graph is characterized through a function (of substitution) that can be self-definable. The purpose of this work is to construct evolution operators for orbit $\{w^k(\mathrm{G})\}$, where each element of $\{w^k(\mathrm{G})\}$ is obtained by substituting each vertex of the previous element by a graph. Here, both the initial graph G as the family of graphs of substitution, are known. In this paper, simple and finite graphs will be used, framed in the graphs theory's area.

Keywords: Graph; distribution operator; substitution of graph; realizable graph; discrete dynamical systems.

2010 Mathematics Subject Classification: 05C10-47N60.

1 Introduction

The graphs to be considered will be simple and finite and with a nonempty set of edges. For a graph G, $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. The cardinality of

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$V(G)$ is called order of G and often it is written as |G|, and the cardinality of $E(G)$ is called size of G and generally it is written as |E(G)|. A (p, q) graph has order p and size q. Two vertices u and v are called neighbors if $\{u, v\}$ is an edge of G. For any vertex v of G, denote by N_v the neighbors of v. Other concepts used in this work and not defined explicitly can be found in the references [1], [2], [3], [4], [5], [6], [7].

2 Preliminares

Some essential concepts of this work are the following:

2.1 Substitution

Let suppose that G and K are two graphs disjointed by vertices. For a non isolated vertex v in $V(G)$ and a function $s : N_v \to V(K)$ it will be defined the substitution of the vertex v by the graph K, as the graph M, denoted by $G(v, s)K$, such that:

(1) $V(M) = (V(G) \cup V(K)) - \{v\}$ and

(2) $E(M) = (E(G) \cup E(K) - \{vx/x \in N_v\}) \cup \{xs(x)/x \in N_v\}$.

It is said that the vertex v is the substituted vertex by K in G under the function s and this function is called substitution function. (See [5], [8]).

If v is isolated, then $M = (G - v) \oplus K$.

Definition 2.1. Let $V(G) = \{v_1, \dots, v_n\}$ and let $H_1, \dots, H_n$ be a sequence of pairwise vertex disjoint graphs. Iteratively, we define the following graphs: $M_0 = G$ and $M_k = M_{k-1}(v_k, s_k)H_k$ the graph which is obtained by substitution of vertices of G by graphs H_i, $1 \le i \le k$.

In other words, M_1, denotes a graph obtained by substitution of only one vertex of G, M_2 denotes a graph obtained by substitution of only one vertex of M_1, and so on. Note that every substituted vertex must belong to $V(G)$.

If v is isolated, then $M = (G - v) \oplus K$. An edge of the substitution graph M_p is an internal edge if it is $s_i(x)s_i(y)$, (see [8]).The edge in M_p that not an internal edge is called an external edge (see [5]). Let G be a graph without isolated vertices, if each vertex v of G is substituted by a complete graph with $val(v)$ vertices, through an injective substitution function, then it will be said that the graph G has been expanded (see [5]). When each vertex of a given G graph is substituted for a copy of G through injective subtitution functions a special type of substitution is obtained which will be called self-substitution and denoted by $G(G)$. See more [8], [9].

2.2 Realizable Graph

A (p, q) graph G is said to be realizable on $\mathbb{R}^3$ if it is possible to distinguish a collection of p different points of $\mathbb{R}^3$, that correspond to the vertices of G and a collection of q curves, pairwise disjoint except, possibly, in extreme points, that correspond to the edges of G such that if a curve γ corresponds to the edge $e = uv$, then only extreme points of γ correspond to vertices of G, namely u and v (More details [2], [3]).

If v is isolated, then $M = (G - v) \oplus K$. In this work, we use the following realization concept. Here, Ω represents the class of all simple and finte graphs. For a set $A \subset \mathbb{R}^3$ and for $\lambda \in \mathbb{R}^+$ the neighborhood of A is defined, denoted by A_λ, as the subset $\mathbb{R}^3$ defined by

$$A_\lambda = \left\{ y \in \mathbb{R}^3 \ / \ \exists x \in A : d(x, y) < \lambda \right\}.$$

Here d is the usual metric on $\mathbb{R}^3$.

Remark 2.1. $A_\lambda = \underset{x\in A}{\cup} V_\lambda(x)$, where $V_\lambda(x) = \left\{y \in \mathbb{R}^3 \ / \ d(x,y) < \lambda\right\}$

Example 2.1. $A = \{(1,2,1)\} \wedge \lambda = 1 \Rightarrow A_1 = V_1\left((1,2,1)\right)$:*Ball of radius 1*

Example 2.2. $A = \{(1,2,1),(2,-2,3)\} \wedge \lambda = \sqrt{2}$, *then:* $A_{\sqrt{2}} = V_{\sqrt{2}}\left((1,2,1)\right) \cup V_{\sqrt{2}}\left(((2,-2,3)\right)$: *Union of two disjoint balls of radius* $\sqrt{2}$.

2.3 Realization of a Graph

If $G \in \Omega$ and $h : V(G) \to \mathbb{R}^3$ is a injective function, then the realization of G in $\mathbb{R}^3$, denoted by G^*, is defined by

$$G^* = \{h(v)/v \in V(G)\} \cup \left(\bigcup_{uv\in E(G)} h(u)h(v)\right),$$

where

$$h(u)h(v) = \{h(u) + t\left(h(v) - h(u)\right)/t \in [0,1]\}.$$

This realization must satisfy the following conditions:

(i) If $G_1, G_2 \in \Omega$ and $G_1 \neq G_2$, then $G_1^* \cap G_2^* = \phi$.

(ii) If $G_1, G_2 \in \Omega$ and $G_1^* \cap G_2^* \neq \Phi$, then $G_1 = G_2$.

(iii) Each $G \in \Omega$ admits one and only one realization G^*. This identification defines the dual class $\Omega^* = \left\{G^* \subset \mathbb{R}^3 \ / \ G \in \Omega\right\}$.

2.4 Distance on Ω

If $G_1, G_2 \in \Omega$, then the distance from G_1 to G_2, denoted by $\mathcal{D}\left(G_1, G_2\right)$, is defined by $\mathcal{D}(G_1,G_2) = Inf\{\lambda \in \mathbb{R}^+/G_1^* \subset (G_2^*)_\lambda \wedge G_2^* \subset (G_1^*)_\lambda\}$

Example 2.3. *If* $G_1^* = \{i,j,k\} \cup ij \cup ik \cup jk$ *and* $G_2^* = \left\{0, \frac{i}{2}\right\} \cup 0\frac{i}{2}$, *then*

$$\mathcal{D}(G_1,G_2) = d\left(\frac{i}{2}, j\right) = \frac{\sqrt{5}}{2}$$

Example 2.4. *If* $G_1^* = \{0\}$ *and*

$G_2^* = \left\{j + mi \ / \ m \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}\right\} \cup \left(\bigcup_{t\in\{1,2,3,4\}} \left(j + \frac{(t-1)i}{4}\right)\left(j + \frac{ti}{4}\right)\right)$, *then* $\mathcal{D}(G_1,G_2) = d(0, j + i) = \sqrt{2}$

2.5 Compactness

If $A \subset \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}^+$, then the compactness of A, denoted by A_ε, is the set $\{y \in \mathbb{R}^3/\exists x \in A, d(x,y) < \epsilon\}$. Here, d is usual distance on $\mathbb{R}^3$. Note that $A \subset A_\varepsilon$ and that A_ε is compact according to usual topology of $\mathbb{R}^3$.

If v is isolated, then $M = (G - v) \oplus K$. For more details *Compactness* , see [6], [7].

2.6 Discrete Dynamical Systems

A discrete dynamical system is any set X together with a mapping $f : X \to X$. In this work X is always a set of graphs. In literature [6], X must have some topology on which f is continuous.

If v is isolated, then $M = (G - v) \oplus K$. For more details *Discrete dynamical systems*, see [10], [11], [6].

2.7 Attractor Points

An orbit of f, $f : \Omega \to \Omega$, being f continuous in topology of Ω, is any sequence of the form $\{G, f(G), ..., f^n(G), ...\}$.

Definition 2.2. A graph G is an *attractor point* of f if there exists some natural number $n > 1$ such that $G = f^n(G)$.

Lemma 2.5. *G^* is compact on $\mathbb{R}^3$.*

Proof. See [9]. □

In the following proposition a distance on Ω will be constructed.

Proposition 2.1. *The function $\mathbf{D} : \Omega \times \Omega \to \mathbb{R}$ defined by $\mathbf{D}(G_1, G_2) = Inf\{\lambda \in \mathbb{R}/G_1^* \subset (G_2^*)_\lambda \wedge G_2^* \subset (G_1^*)_\lambda\}$, is a distance on Ω.*

Proof. See [9]. □

Theorem 2.6. *$(\Omega, \mathbf{D})$ is a complete metric space.*

Proof. See [9]. □

Now, the following lemma is fundamental.

Lemma 2.7. *If w is a function of Ω in Ω, defined by $w(G) = M_p(G)$, where p is the order of G, then w is a contraction in Ω.*

Proof. See [9]. □

The existence of an attractor point to w is assured by the following theorem whose demonstration could be found in [12].

Theorem 2.8. *If M is a complete metric space andf $: M \to M$ is a contraction, then there exists $x_f = \lim_{n \longrightarrow \infty} f^n(x)$. Moreover, x_f is independent of the choice of x in M. Also x_f is the only fixed point of f.*

Theorem 2.9. *If $G \in \Omega$, then the orbit $G \longrightarrow w(G) \longrightarrow ...w^k(G) \longrightarrow ...$ has a single attractor point G_w for w.*

Proof. Apply Lemma 2.7 and Theorem 2.8. □

See more [1], [7].

3 Evolution Operator

In this Section we introduce the concept of evolution operator and analyze its behavior. A evolution operator of bipartite complete graphs is a pair (γ, λ) where γ is a continuous function (in the topology induced by the distance $\mathbf{D}$ defined in the Theorem 2.6) of Ω in Ω that choose bipartite complete graph of Ω and λ is a family $\langle s_1, ..., s_p\rangle$ of functions of substitution relating to each bipartite complete preimage of the function γ such that if G is a bipartite complete graph $K(r, s)$ then $\gamma(K(r, s)) = M_p(K(r, s))$.This operator is constructed through substitution, as it follows:

If v is isolated, then $M = (G - v) \oplus K$. If $v_1, ..., v_p$ are the vertices of a bipartite complete graph G and $H_1, ..., H_p$ is a sequence of graph with no vertices in common between them or with $K(r, s)$

then $M_p(K(r,s))$ will denoted the graph obtained by substitution of p vertices of $K(r,s)$ by graphs H_i, $1 \leq i \leq p$, where $M_0(K(r,s)) = K(r,s)$, $M_1(K(r,s))$, will denoted the graph obtained by substitution of only a vertex of $K(r,s)$ through of a inyective function of substitution s_1, $M_2(K(r,s))$ will denoted the graph obtained by substitution of only a vertex of $M_1(K(r,s))$ through of a through of a inyective function of substitution s_2, and so forth. Note that each substituted vertex must belong to $V(K(r,s))$. Here the family $(s_1, ..., s_p)$ of substitution functions determine λ and w determine γ, where w is the continuous function built in the Lemma 2.7. Moreover, if the order of each H_i is p_i, then the order of $M_p(K(r,s))$ is $\sum_{i=1}^{p} p_i$. Succesively the rest of the elements of the orbit $\{w^k(K(r,s))\}$ are obtained.

Definition 3.1. The orbit $\{w^k(K(r,s))\}$ is said to be complete bipartite if each substitution function is injective.

In order to illustrate the ideas above then we will give the following examples:

Example 3.1. *Let* $G = K(1,4), V(K(1,4), H_1 = K_4, H_i = K_1, 2 \leq i \leq 5$. *be. In that manner, if* $V(K(1,4)) = \{v_1, v_2, v_3, v_4, v_5\}$ *with bipartition* $A = \{v_1\}$ *and* $B = \{v_2, v_3, v_4, v_5\}$ *with* $\lambda = (s_1, ..., s_5), s_i : N_{v_i} \to V(H_{deg(v_i)}), 1 \leq i \leq 5$, *injective function. Then* $w(K(1,4)) = M_8(K(1,4))$

In Fig. 1 the operator action is illustred.

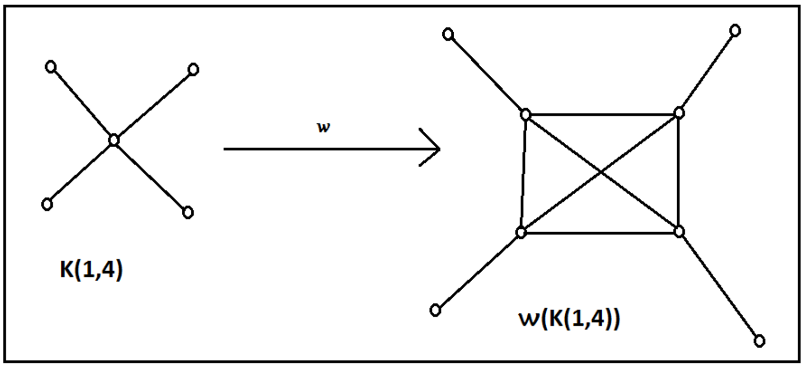

Fig. 1. Operator action

Example 3.2. *Let* $G = K(2,3)$ *and* $H_i, 1 \leq i \leq 5$, *be a family of complete graphs, disjoint between them and with* G, *such that if* $v \in V(K(2,3))$ *then* $H_i \equiv K_{deg(v)}$*(where* $\equiv$ *indicates graphs isomorphism). That way, if* $V(K(2,3)) = \{v_1, v_2, v_3, v_4, v_5\}$ *with bipartition* $A = \{v_1, v_2\}$ *and* $B = \{v_3, v_4, v_5\}$ *with* $\lambda = (s_1, ..., s_5), s_i : N_{v_i} \to V(H_{deg(v_i)}), 1 \leq i \leq 5$, *injective function. Then* $w(K(2,3)) = M_{12}(K(2,3))$

If v *is isolated, then* $M = (G - v) \oplus K$. *In Fig. 2 the operator action is illustrated.*

The following theorems, in some way, characterize complete bipartite evolution operators.

Theorem 3.3. *If* $\{w^k(K(m,p-m))\}, m < p$, *is a complete bipartite orbit, then for each* k , $n_c\left(w^k(K(m,p-m)\right) = \max\{m, p-m\}$.

Proof. Suppose that $n_c\left(w^k(K(m,p-m))\right) = \max\{m, p-m\} = m$. By applying the evolution operator to state $w^k(K(m,p-m))$ a complete subgraph of order m is obtained. As the rest of the blocks of $w^{k+1}(K(m,p-m))$ has order m or $p-m$, we have that $n_c\left(w^{k+1}(K(m,p-m))\right) = m$. Analogously if $n_c\left(w^k(K(m,p-m))\right) = \max\{m, p-m\} = p-m$. $\square$

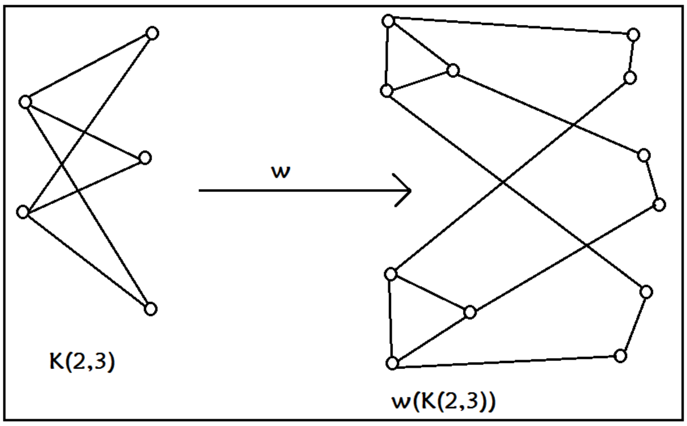

Fig. 2. Operator action

Theorem 3.4. *If* $\{w^k(K(m,p-m))\}, m < p$, *is a complete bipartite orbit, then for each* k ,

$$d_{HB}\left(w^k(K(m,p-m)\right) = \frac{\ln\left(\sum\limits_{v\in V(w^{k-1}(K(m,p-m))}\deg(v)\right)}{\ln\left(|w^{k-1}(K(m,p-m)|\right)} ,$$

where d_{HB} *is the Hausdorff-Besicovitch dimension.*

Proof. We have that:

$$|w(K(m,p-m))| = \sum_{v\in V((K(m,p-m))}\deg(v).$$

in the same way:

$$\left|w^2(K(m,p-m))\right| = \sum_{v\in V(w((K(m,p-m)))}\deg(v).$$

Generalizing, we have:

$$\left|w^k(K(m,p-m))\right| = \sum_{v\in V(w^{k-1}(K(m,p-m)))}\deg(v).$$

Therefore,

$$d_{HB}(\left(w^k(K(m,p-m))\right)) = \frac{\ln\left(\sum\limits_{v\in V(w^{k-1}(K(m,p-m)))}\deg(v)\right)}{\ln\left(|w^{k-1}(K(m,p-m))|\right)}$$

□

4 Conclusions

The present proposal presents a generalization of the concept of substitution [9], allowing the approach of hypotheses in relation to infinite processes [13] and to enrich the asymptotic properties of such discrete dynamical systems [11] and its relationship with the fractal dimensions [6], through the definition of Hausdorff-Besicovitch dimension [14], [15] in this context.

Acknowledgement

The authors want to thank the Department of Mathematics and Statistics, Universidad de Playa Ancha, for the provided support when carrying out this work.

Work was supported and financed by DGI of Universidad de Playa Ancha, through the 01-1314 project.

Competing Interests

The authors declare that no competing interests exist.

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