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Solution of Nonlinear Space–time Fractional Differential Equation via the Triple Fractional Riccati Expansion Method

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Abstract

In this paper, the triple fractional Riccati expansion method is applied to solve fractional differential equation. To illustrate the effectiveness of the method, the nonlinear space-time fractional Klein–Gordon equation is studied. The obtained solutions include generalized trigonometric and hyperbolic function solutions. Among these solutions, some are found for the first time.

Keywords: Triple fractional Riccati expansion method, nonlinear fractional differential equation, modified Riemann–Liouville derivative, exact solution, nonlinear space-time fractional Klein–Gordon equation.

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1 Introduction

Many phenomena in physics, engineering, biology, chemistry, finance and other areas of applications are described by nonlinear fractional differential equations (FDEs). The fractional order partial differential is the generalizations of classical integer order partial differential equations. Fractional derivatives provide an excellent instrument for the description of memory properties of various processes. This is the main advantage of fractional derivatives in comparison with classical integer- order models, in which such effects are in fact neglected [1- 7]. Searching for analytical and numerical solutions of FDEs is currently a very active area of research. In the past two decades, both mathematicians and physicists have made much significant work in this direction and presented some effective methods. Examples include: the Laplace transform method, the Fourier transform method, the iteration method, the operational method, finite difference method, finite element method, Adomian decomposition method, differential transform method, variational iteration method, homotopy perturbation method, the fractional sub-equation method, and generalized fractional sub-equation method [8-33]. Recently, Abdel-Salam and Yousif [21] introduced the fractional Riccati expansion method by solving the fractional differential equation $D_{\xi}^{\alpha} F = A + B F^2, \qquad 0 < \alpha \le 1,$ to obtain analytical solutions of FDEs with constant coefficients. They solved the Space-time fractional KdV equation, regularized long-wave equation, Boussinesq equation and Klein–Gordon equation. In this paper, we generalized this method by introducing the triple fractional Riccati expansion method to obtain many exact travelling wave solutions of nonlinear FDEs with the Jumarie's modified Riemann–Liouville derivative [22–23].

This paper is organized as follows: the description of the triple fractional Riccati expansion method is presented in section 2. In section 3, the solution of the space-time fractional Klein-Gordon equation is studied. In section 4, discussion and conclusion are presented.

2 Jumarie's Modified Riemann–Liouville Derivative and Triple Fractional Riccati Expansion Method

The Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression [22, 23]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi) - f(0)]d\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi) - f(0)]d\xi, & 0 < \alpha < 1 \\ [f^{(\alpha-n)}(x)]^{(n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{1}$$

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative were summarized in [23]. Three of them (which will be used in the following sections) are

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \qquad \gamma > 0, \tag{2}$$

$$D_x^\alpha [f(x)g(x)] = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{3}$$

$$D_x^\alpha f[g(x)] = f_g^{'}[g(x)] D_x^\alpha g(x) = D_g^\alpha f[g(x)](g_x^{'})^\alpha. \tag{4}$$

We outline the main steps of the triple fractional Riccati expansion method for solving FDEs. For a given nonlinear FDE, say, in two variables x and t

$$P(u,\ D_t^\alpha u, D_x^\alpha u,\ D_t^{2\alpha} u,\ D_x^{2\alpha} u,\ \ldots) = 0, \tag{5}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann–Liouville derivatives of u, $u = u(x,\ t)$ is unknown function, P is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 1. By using the travelling wave transformation:

$$u(x,\ t) = u(\xi), \qquad \xi = x + \omega t, \tag{6}$$

where ω is a constant to be determined later, the nonlinear FDE (5) is reduced to the following nonlinear fractional ordinary differential equation (FODE) for $u = u(\xi)$:

$$\tilde{P}(u,\ \omega^\alpha D_\xi^\alpha u, D_\xi^\alpha u,\ \omega^{2\alpha} D_\xi^{2\alpha} u,\ D_\xi^{2\alpha} u,\ \ldots) = 0, \tag{7}$$

Step 2. We suppose that $u(\xi)$ can be expressed as

$$u(\xi) = a_0 + \sum_{i=1}^{n} h^{i-1}(a_i\, f + b_i g + c_i h), \tag{8}$$

where $a_0,\ a_i,\ b_i,\ c_i$ are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in equation (5) or equation (7) and $f = f(\xi),\ g = g(\xi),\ h = h(\xi)$ satisfies the following fractional Riccati equations:

$$\begin{aligned}
&D_\xi^\alpha f = \delta[kg(1 - 2Bg - 2h - 2\varepsilon Ah) + B f^2],\\
&D_\xi^\alpha g = \delta f(1 - Bg - 2h),\\
&D_\xi^\alpha h = \delta f(2Ag + Bh), \qquad 0 < \alpha \le 1,\\
&Af^2 = kh(1 - Bg - h - \varepsilon Ah),\\
&Ag^2 = h(1 - Bg - h).
\end{aligned} \tag{9}$$

where $B,\ A \neq 0$ are arbitrary constants and $k = \pm 1,\ \delta = \pm 1, \varepsilon = \pm 1$. Using the Mittag-Leffler function in one parameter $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \quad (\alpha > 0)$, we obtain the following solution of equation (9):

Case 1: If $k = 1,\ \delta = -1,$ and $\varepsilon = 1,$ then (9) has the solution

$$\begin{aligned} f(\xi) &= \frac{\sinh(\xi,\ \alpha)}{1 + B\cosh(\xi,\ \alpha) + A\cosh^2(\xi,\ \alpha)}, \\ g(\xi) &= \frac{\cosh(\xi,\ \alpha)}{1 + B\cosh(\xi,\ \alpha) + A\cosh^2(\xi,\ \alpha)}, \\ h(\xi) &= \frac{1}{1 + B\cosh(\xi,\ \alpha) + A\cosh^2(\xi,\ \alpha)}. \end{aligned} \tag{10}$$

Case 2: If $k = 1,\ \delta = -1,$ and $\varepsilon = -1,$ then (9) has the solution

$$\begin{aligned} f(\xi) &= \frac{\cosh(\xi,\ \alpha)}{1 + B\sinh(\xi,\ \alpha) + A\sinh^2(\xi,\ \alpha)}, \\ g(\xi) &= \frac{\sinh(\xi,\ \alpha)}{1 + B\sinh(\xi,\ \alpha) + A\sinh^2(\xi,\ \alpha)}, \\ h(\xi) &= \frac{1}{1 + B\sinh(\xi,\ \alpha) + A\sinh^2(\xi,\ \alpha)}. \end{aligned} \tag{11}$$

Case 3: If $k = -1,\ \delta = 1,$ and $\varepsilon = 1,$ then (9) has the solution

$$\begin{aligned} f(\xi) &= \frac{\sin(\xi,\ \alpha)}{1 + B\cos(\xi,\ \alpha) + A\cos^2(\xi,\ \alpha)}, \\ g(\xi) &= \frac{\cos(\xi,\ \alpha)}{1 + B\cos(\xi,\ \alpha) + A\cos^2(\xi,\ \alpha)}, \\ h(\xi) &= \frac{1}{1 + B\cos(\xi,\ \alpha) + A\cos^2(\xi,\ \alpha)}. \end{aligned} \tag{12}$$

Case 4: If $k = -1,\ \delta = -1,$ and $\varepsilon = 1,$ then (9) has the solution

$$f(\xi)=\frac{\cos(\xi,\,\alpha)}{1+B\sin(\xi,\,\alpha)+A\sin^2(\xi,\,\alpha)},$$
$$g(\xi)=\frac{\sin(\xi)}{1+B\sin(\xi,\,\alpha)+A\sin^2(\xi,\,\alpha)}, \tag{13}$$
$$h(\xi)=\frac{1}{1+B\sin(\xi,\,\alpha)+A\sin^2(\xi,\,\alpha)},$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\cosh(\xi,\alpha)=\frac{E_\alpha(\xi^\alpha)+E_\alpha(-\xi^\alpha)}{2}, \qquad \sinh(\xi,\alpha)=\frac{E_\alpha(\xi^\alpha)-E_\alpha(-\xi^\alpha)}{2},$$
$$\cos(\xi,\alpha)=\frac{E_\alpha(i\xi^\alpha)+E_\alpha(-i\xi^\alpha)}{2}, \qquad \sin(\xi,\alpha)=\frac{E_\alpha(i\xi^\alpha)-E_\alpha(-i\xi^\alpha)}{2i}. \tag{14}$$

Step 3. Substituting the fractional Riccati expansion method (8) into the FODE (7), then the left-hand side of equation (7) can be converted into a polynomial in $f^i g^j h^l$ $i=0,\,1,\;j=0,\,1,\;l=0,1,2,3,...$. Setting each coefficient of the polynomial to zero yields system of algebraic equations for $a_0,\,a_1,\;...,\,a_n,b_1,\;...,\,b_n,c_1,\;...,\;c_n$, and ω.

Step 4. By solving the system obtained in step 3, the constant $a_0,\,a_1,\;...,\,a_n$, $b_1,\;...,\,b_n,c_1,\;...,\;c_n$,, and ω can be expressed by the parameters A and B. Depending on the chosen values of k, δ, and ε the functions $f(\xi)$, $g(\xi)$, $h(\xi)$ possesses the travelling wave solutions as given above; then the fractional triple Riccati expansion method (8) has the travelling wave solution of the nonlinear FDEs (5).

Remark 1. When $\alpha=1$ equation (9) becomes equation (1); see [24].

Remark 2 It can be easily found that if $\alpha=1$, and $B=0$, then equation (9) becomes equation (6). For more details see [25].

3 Space-time Fractional Klein–Gordon Equation

The nonlinear Klein–Gordon equation appears as a model of self-focusing waves in nonlinear optics [26] and its physical motivation is present in various branches of physics; examples include plasma physics and fluid mechanics. In addition to, it is relativistic quantum equation for particles with zero spin [27, 28]. The nonlinear space-time fractional Klein–Gordon equation, which is a transformed generalization of the nonlinear Klein–Gordon equation, is defined as follows:

$$D_t^{2\alpha}u-D_x^{2\alpha}u+\mu u-\tau u^3=0, \qquad 0<\alpha\le 1, \tag{15}$$

where $u = u(x,t)$, μ , τ are arbitrary constants and α is the fractional order derivative.

By using the travelling wave transformation $u(x,\, t) = u(\xi),\ \xi = x + \omega t$, where ω is the dimensionless velocity of the wave, then, (15) is reduced to the following nonlinear FODEs:

$$(\omega^{2\alpha} - 1)D_\xi^{2\alpha} u + \mu u - \tau u^3 = 0. \tag{16}$$

Balancing $D_\xi^{2\alpha} u$ with u^3 gives $n = 1$. Therefore, the solution of equation (16) can be expressed as

$$u = a_0 + a_1 f(\xi) + b_1 g(\xi) + c_1 h(\xi). \tag{17}$$

Substituting (17) into (16) using (9) and setting the coefficients of $f^i g^j h^l$ to zero, we obtain system of algebraic equations for $a_0,\ a_1,\ b_1,\ c_1$, and ω. Solving this system, we obtain the following cases:

Case 1:

$$k = 1,\ \delta = -1,\ \varepsilon = 1,\ \ c_1 = \sqrt{\frac{2\mu}{\tau}},\ \ B = a_0 = a_1 = b_1 = 0,\ \ A = -2,\ \ \omega^{2\alpha} = 1 - \frac{\mu}{4}. \tag{18}$$

Case 2:

$$k = 1,\ \delta = -1,\ \varepsilon = 1,\ \ a_1 = \sqrt{-\frac{2\mu}{\tau}},\ \ B = a_0 = b_1 = c_1 = 0,\ \ A = -1,\ \ \omega^{2\alpha} = 1 - \mu. \tag{19}$$

Case 3:

$$k = 1,\ \delta = -1,\ \varepsilon = 1,\ a_1 = \sqrt{\frac{2A\mu}{\tau}},\ \ b_1 = \sqrt{\frac{2A\mu(1+A)}{\tau}},\ \ B = a_0 = c_1 = 0,\ \omega^{2\alpha} = 1 - \mu. \tag{20}$$

Case 4:

$$k = 1,\ \delta = -1,\ \varepsilon = -1,\ \ c_1 = \sqrt{\frac{2\mu}{\tau}},\ \ B = a_0 = a_1 = b_1 = 0,\ \ A = 2,\ \ \omega^{2\alpha} = 1 - \frac{\mu}{4}. \tag{21}$$

Case 5:

$$k = 1,\ \delta = -1,\ \varepsilon = -1,\ a_1 = \sqrt{\frac{2\mu}{\tau}},\ \ B = a_0 = b_1 = c_1 = 0,\ \ A = 1,\ \ \omega^{2\alpha} = 1 - \mu. \tag{22}$$

Case 6:

$$k = 1,\ \delta = -1,\ \varepsilon = -1,\ a_1 = \sqrt{\frac{2A\mu}{\tau}},\ b_1 = \sqrt{\frac{2A\mu(1-A)}{\tau}},\ B = a_0 = c_1 = 0,\ \omega^{2\alpha} = 1 - \mu. \tag{23}$$

Case 7:

$$k = -1,\ \delta = 1,\ \varepsilon = 1,\ \ c_1 = \sqrt{\frac{2\mu}{\tau}},\ B = a_0 = a_1 = b_1 = 0,\ \ A = -2,\ \ \omega^{2\alpha} = 1 + \frac{\mu}{4}. \tag{24}$$

Case 8:

$$k = -1,\ \delta = 1,\ \varepsilon = 1,\ a_1 = \sqrt{\frac{2\mu}{\tau}},\ \ B = a_0 = b_1 = c_1 = 0,\ \ A = -1,\ \ \omega^{2\alpha} = 1 + \mu. \tag{25}$$

Case 9:

$$k=-1,\ \delta=1,\ \varepsilon=1,\ a_1=\sqrt{-\frac{2A\mu}{\tau}},\ b_1=\sqrt{\frac{2A\mu(1+A)}{\tau}},\ B=a_0=c_1=0,\ \omega^{2\alpha}=1+\mu. \quad (26)$$

Case 10:

$$k=-1,\ \delta=-1,\ \varepsilon=1,\ \ c_1=\sqrt{\frac{2\mu}{\tau}},\ B=a_0=a_1=b_1=0,\ A=-2,\ \omega^{2\alpha}=1+\frac{\mu}{4}. \quad (27)$$

Case 11:

$$k=-1,\ \delta=-1,\ \varepsilon=1,\ a_1=\sqrt{\frac{2\mu}{\tau}},\ \ B=a_0=b_1=c_1=0,\ A=-1,\ \omega^{2\alpha}=1+\mu. \quad (28)$$

Case 12:

$$k=-1,\ \delta=-1,\ \varepsilon=1,\ a_1=\sqrt{-\frac{2A\mu}{\tau}},\ b_1=\sqrt{\frac{2A\mu(1+A)}{\tau}},\ B=a_0=c_1=0,\ \omega^{2\alpha}=1+\mu. \quad (29)$$

Therefore, the analytical solutions of the space-time fractional Klein–Gordon equation are

$$u_1=\sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\cosh^2(x+\omega t,\ \alpha)}\right], \qquad \omega=\left[1-\frac{\mu}{4}\right]^{\frac{1}{2\alpha}}, \quad (30)$$

$$u_2=\sqrt{-\frac{2\mu}{\tau}}\ \operatorname{csch}(x+\omega t,\ \alpha), \qquad \omega=[1-\mu]^{\frac{1}{2\alpha}}, \quad (31)$$

$$u_3=\sqrt{\frac{2A\mu}{\tau}}\left[\frac{\sinh(x+\omega t,\ \alpha)}{1+A\cosh^2(x+\omega t,\ \alpha)}+\frac{\sqrt{1+A}\ \cosh(x+\omega t,\ \alpha)}{1+A\cosh^2(x+\omega t,\ \alpha)}\right], \qquad \omega=[1-\mu]^{\frac{1}{2\alpha}}, \quad (32)$$

$$u_4=\sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1+2\sinh^2(x+\omega t,\ \alpha)}\right], \qquad \omega=\left[1-\frac{\mu}{4}\right]^{\frac{1}{2\alpha}}, \quad (33)$$

$$u_5=\sqrt{\frac{2\mu}{\tau}}\ \operatorname{sech}(x+\omega t,\ \alpha), \qquad \omega=[1-\mu]^{\frac{1}{2\alpha}}, \quad (34)$$

$$u_6=\sqrt{\frac{2A\mu}{\tau}}\left[\frac{\cosh(x+\omega t,\ \alpha)}{1+A\sinh^2(x+\omega t,\ \alpha)}+\frac{\sqrt{1-A}\ \sinh(x+\omega t,\ \alpha)}{1+A\sinh^2(x+\omega t,\ \alpha)}\right], \qquad \omega=[1-\mu]^{\frac{1}{2\alpha}}, \quad (35)$$

$$u_7=\sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\cos^2(x+\omega t,\ \alpha)}\right], \qquad \omega=\left[1+\frac{\mu}{4}\right]^{\frac{1}{2\alpha}}, \quad (36)$$

$$u_8 = \sqrt{\frac{2\mu}{\tau}}\ \csc(x+\omega t,\ \alpha), \qquad \omega = [1+\mu]^{\frac{1}{2\alpha}}, \tag{37}$$

$$u_9 = \sqrt{-\frac{2A\mu}{\tau}}\left[\frac{\sin(x+\omega t,\ \alpha)}{1+A\cos^2(x+\omega t,\ \alpha)} + \frac{\sqrt{-(1+A)}\ \cos(x+\omega t,\ \alpha)}{1+A\cos^2(x+\omega t,\ \alpha)}\right], \quad \omega = [1+\mu]^{\frac{1}{2\alpha}}, \tag{38}$$

$$u_{10} = \sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\sin^2(x+\omega t,\ \alpha)}\right], \qquad \omega = \left[1+\frac{\mu}{4}\right]^{\frac{1}{2\alpha}}, \tag{39}$$

$$u_{11} = \sqrt{\frac{2\mu}{\tau}}\ \sec(x+\omega t,\ \alpha), \qquad \omega = [1+\mu]^{\frac{1}{2\alpha}}, \tag{40}$$

$$u_{12} = \sqrt{-\frac{2A\mu}{\tau}}\left[\frac{\cos(x+\omega t,\ \alpha)}{1+A\sin^2(x+\omega t,\ \alpha)} + \frac{\sqrt{-(1+A)}\ \sin(x+\omega t,\ \alpha)}{1+A\sin^2(x+\omega t,\ \alpha)}\right], \quad \omega = [1+\mu]^{\frac{1}{2\alpha}}. \tag{41}$$

When $\alpha = 1$, we obtain the nonlinear Klein–Gordon equation

$$u_{tt} - u_{xx} + \mu u - \tau u^3 = 0, \tag{42}$$

as special case of equation (15). Solutions given in equations (30) - (41) are reduced to the following solutions of the Klein-Gordon equation (42) [24]

$$u_{1KG} = \sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\cosh^2(x+\omega t)}\right], \qquad \omega = \frac{1}{2}\sqrt{4-\mu},, \tag{43}$$

$$u_{2KG} = \sqrt{-\frac{2\mu}{\tau}}\ \operatorname{csch}(x+\omega t), \qquad \omega = \sqrt{1-\mu}, \tag{44}$$

$$u_{3KG} = \sqrt{\frac{2A\mu}{\tau}}\left[\frac{\sinh(x+\omega t)}{1+A\cosh^2(x+\omega t)} + \frac{\sqrt{1+A}\ \cosh(x+\omega t)}{1+A\cosh^2(x+\omega t)}\right], \qquad \omega = \sqrt{1-\mu}, \tag{45}$$

$$u_{4KG} = \sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1+2\sinh^2(x+\omega t)}\right], \qquad \omega = \frac{1}{2}\sqrt{4-\mu}, \tag{46}$$

$$u_{5KG} = \sqrt{\frac{2\mu}{\tau}}\ \operatorname{sech}(x+\omega t), \qquad \omega = \sqrt{1-\mu}, \tag{47}$$

$$u_{6KG}=\sqrt{\frac{2A\mu}{\tau}}\left[\frac{\cosh(x+\omega t)}{1+A\sinh^2(x+\omega t)}+\frac{\sqrt{1-A}\ \sinh(x+\omega t)}{1+A\sinh^2(x+\omega t)}\right],\qquad \omega=\sqrt{1-\mu},\quad (48)$$

$$u_{7KG}=\sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\cos^2(x+\omega t,\,\alpha)}\right],\qquad \omega=\frac{1}{2}\sqrt{4+\mu},\quad (49)$$

$$u_{8KG}=\sqrt{\frac{2\mu}{\tau}}\ \csc(x+\omega t),\qquad \omega=\sqrt{1+\mu},\quad (50)$$

$$u_{9KG}=\sqrt{-\frac{2A\mu}{\tau}}\left[\frac{\sin(x+\omega t)}{1+A\cos^2(x+\omega t)}+\frac{\sqrt{-(1+A)}\ \cos(x+\omega t)}{1+A\cos^2(x+\omega t)}\right],\qquad \omega=\sqrt{1+\mu},\quad (51)$$

$$u_{10KG}=\sqrt{\frac{2\mu}{\tau}}\left[\frac{1}{1-2\sin^2(x+\omega t)}\right],\qquad \omega=\frac{1}{2}\sqrt{4+\mu},\quad (52)$$

$$u_{11KG}=\sqrt{\frac{2\mu}{\tau}}\ \sec(x+\omega t),\qquad \omega=\sqrt{1+\mu},\quad (53)$$

$$u_{12KG}=\sqrt{-\frac{2A\mu}{\tau}}\left[\frac{\cos(x+\omega t)}{1+A\sin^2(x+\omega t)}+\frac{\sqrt{-(1+A)}\ \sin(x+\omega t)}{1+A\sin^2(x+\omega t)}\right],\qquad \omega=\sqrt{1+\mu}\,.\quad (54)$$

4 Discussion and Conclusion

The triple fractional Riccati expansion method has been used to construct more types of analytical solutions of the space-time fractional Klein–Gordon equation. We have obtained many solutions including generalized hyperbolic and trigonometric functions. With the best of our knowledge some of these solutions are obtained for the first time. Possible applications of the results obtained in this paper are in mathematics and physics. Also, we are investigating how our method can be modified to treat higher dimensional FDEs, vector non-linear fractional evolution equations, and other FDEs of a different kind.

Competing Interests

Authors have declared that no competing interests exist.

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